<< *Research eBook* >>

PSEUDO-CONFORMAL FIELD THEORY

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PREFACE

General relativity and quantum field theory are the successful fundamental theories of the 20th century. The experimental verifications of both theories are so many that no serious question of their validity actually exists. The fundamental mathematical notion of general relativity is the lorentzian metric of the riemannian geometry. It explains very well gravitation but "considers" all the other fields of electromagnetic, weak and strong interactions as external. All the efforts to incorporate them have failed. On the other hand, the fundamental mathematical notion of quantum field theory is the rigged Hilbert space formulated in the general realm of generalized functions (Schwartz distributions). Its last success is the standard model. But the theoretical efforts to generalize it to grand unified and supersymmetric models have not been experimentally verified. The recent LHC experimental result and the failed dark matter searches strongly indicate that supersymmetric particles do not exist. These experimental results drifted to failure string theory, which was the most serious effort to unify general relativity with quantum field theory. The present 4-dimensional lagrangian model consists of a renormalizable generally covariant action based on a special totally real Cauchy-Riemann (CR-) structure. It does not need supersymmetry to incorporate the fermions, which are just distributional solitons.

The euclidean 2-dimensional conformal field theories have been very successful in condensed matter. The Polyakov action is a 2-dimensional conformal field theory based on the 2-dimensional lorentzian metric. In 1986, during my summer stay at CERN, I realized that the "beauty" and the impressive properties of the 2-dimensional conformal field theories come from their metric independence **without being topological**. That is, the fact that any 2-dimensional metric admits coordinates (the light-cone coordinates $(z^0, z^{\widetilde{0}})$) such that $g_{\mu\nu}dx^\mu dx^\nu = g_{0\widetilde{0}}dz^0 dz^{\widetilde{0}}$. Hence, I looked and found a metric independent 4-dimensional action with 4-dimentional metrics, which take the form $g_{\mu\nu}dx^\mu dx^\nu = g_{\alpha\widetilde{\beta}}dz^\alpha dz^{\widetilde{\beta}}$ with $\alpha, \beta = 0, 1$ in a generally complex coordinate system. Not all the 4-dimensional metrics can take this form. Only those which admit two geodetic and shear-free null congruences take this form, like the black-hole metrics. I even considered it as an explanation of the fact that only these solutions of the Einstein equations are observed in nature. The metric independence of the 4-dimensional generally covariant action assures its (formal) renormalizability, simply because it is dimensionless and the geometrical counterterms are not permitted.

The present 4-dimensional action has no other physical relation with string theory. It depends on a special totally real Cauchy-Riemann structure (called lorentzian Caucy-Riemann (LCR-) structure) and it contains a compatible $SU(N)$ gauge field connection. It essentially takes the place of the scalar field $X^\mu(\tau, \sigma)$ of the 2-dimensional conformal action, which string theory interprets as the immersion field of the 2-dimensional surfaces in the 26-dimensional spacetime. The $SU(N)$ gauge field is finally identified with the gluon field. Graviton and photon naturally emerge from the fundamental lorentzian CR-structure (LCR-structure). Electron and its neutrino are the stable static and stationary solitonic LCR-manifolds, which are directly related to an irreducible and reducible quadric of $CP(3)$. They are the massive and massless ruled surfaces with Hopf invariant one, which may be assumed to be the electronic leptonic number. The gyromagnetic ratio of the electron LCR-structure is $g = 2$ even at the classical solitonic level, already computed by Carter for the Kerr-Newman metric. The other generations of fermionic leptons may be those with higher Hopf invariants. The number of the leptonic generations may be restricted to three by the simple fact that the non conformally flat metrics admit no more than four geodetic and shear-free null congruences (Petrov classification). No other leptons exist, up to type III LCR-structure. For every leptonic LCR-manifold, there is a solitonic colored $(SU(N))$ configuration, which are interpreted as the corresponding quarks. The "electron" quark is explicitly derived. This derivation-correspondence makes apparent why the quarks are colored copies of leptons. Hence the present action is apparently the extension of general relativity based on a special totally real CR-structure properly defined in the Cartan moving frame (independent tangent vectors) formalism. The most shocking difference with riemannian geometry is that the Hawking-Penrose singularity theorems do not apply and the Penrose censorship hypothesis is not valid, because the elementary fermionic particles have gravitational dressings with naked ring singularities. That is, the stars are aggregations of naked ring singularities, which are well defined in the context of lorentzian CR-structure.

The present theory is called pseudo-conformal field theory (PCFT) following the initial term for the CR-structure, used by E. Cartan, Tanaka, Severi and others, who first worked on this mathematical notion. Besides, this term is compatible with the name of 2-dimensional conformal field theories used in physics. But the 4-dimensional PCFT is invariant under tetrad-Weyl transformations, which is larger than the metric-Weyl symmetry of the quadratic Weyl-tensor. This tetrad-Weyl symmetry is broken (even at the classical level) by the existence of the conserved quantities of charge and energy-momentum. Hence in brief, LCR-structure is the fundamental structure that replaces the lorentzian riemannian structure of general relativity and PCFT is essentially the lagrangian that Einstein was searching to extend his theory of relativity. On the other hand the Polyakov action is the corresponding 2-dimensional PCFT action. Besides, the mathematics are algebraically based on surfaces of $CP(3)$ in analogy to the well known dependence of string theory on curves of CP^2.

The standard model is derived through the causal perturbative theory of Stuckelber and Bogoliubov combined with the Epstein-Glaser remark and the operational algorithm of Scharf and collaborators, viewed as a direct application of the properties of the Schwartz distributions. The starting point is the Poincaré covariance of the distributional solitons viewed as elements of the rigged Hilbert-Fock space of the free fields tempered distributions. The "internal" $U(2)$ connection is derived from the LCR-tetrad as a Cartan lift, which is directly related to the corresponding geodetic and shear-free null tetrad of Einstein's gravity. Their sources are the leptonic particles identified with the singular supports of the LCR-structures. The pair of the massive electron and its massless neutrino are the massive and massless (developable) quadratic surfaces of $CP(3)$. The relations of the coupling constants and the masses are the necessary conditions for the existence of the product of the tempered distributions and the elimination of the negative norm states, which appear in the S-matrix of the causal perturbative field theory as already described by Scharf in his books.

Hence, PCFT essentially describes the consequences of the simple consideration of the LCR-structure as fundamental geometrical structure, instead of the Einstein metric. I think the most surprising (to me) results are the intimate relation between the electroweak gauge fields with the geodetic and shear-free null tetrad of the Einstein metric and the possibility to "derive" quantum mechanics from the distributional nature of the solitonic LCR-structures. The purpose of the present "Research eBook" is to provide the interested researcher with all the details of the present status of PCFT, essentially following their historical evolution.

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Part I

PSEUDO-CONFORMAL ACTIONS

Synopsis

The wonderful properties of the 2-dimensional Polyakov action are essentially based to its metric independence (without being topological), when written in the 2-dimensional light-cone coordinates. Because of the conformal anomaly this occurs in the 26-dimentional embedding $\Psi^j(\tau)$ of the 2-dimensional string surface. For that, I had to clarify the formulation of the fundamental structure that had to replace the Einstein lorentzian metric of the riemannian structure, because a general 4-dimensional lorentzian metric does not always take the off-diagonal form, that the 2-dimensional lorentzian metric does. In the context of general relativity, Flaherty had studied this kind of 4-dimensional metrics, characterized by admitting two geodetic and shear-free null congruences. The E. Cartan formalism permits the metric independent definition of this fundamental structure, which I call lorentzian Cauchy-Riemann (LCR-) structure, and which is formally invariant under a tetrad-Weyl transformation, replacing the metric-Weyl transformation of the Polyakov action. I write down the covariant 4-dimensional action and I quantize it. As expected, it is dimensionless and formally renormalizable. The lowest order 1-loop diagrams have been computed. The place of the embedding field $\Psi^j(\tau)$ now takes a $SU(N)$ gauge field-like $A_{j\mu}(x)$, which will be identified with the gluon field. But I have not yet found any tetrad-Weyl anomaly that could restrict the order N of the gauge field group, like it happens in Polyakov action. My research steps of this passage from the 2-dimensional Polyakov action to the 4-dimensional generally covariant action, compatible with the LCR-structure, are described in this first part of the Research eBook.

1 THE 2-D LCR-MANIFOLD

The simplest lorentzian CR-manifold M is a real 2-dimensional manifold defined as a real surface of a 2-dimensional complex manifold $\widehat{M}$ with the real equations

$$\begin{aligned}
&\rho_1(\overline{z^0}, z^0) = 0 \quad , \quad \rho_2(\overline{z^{\widetilde{0}}}, z^{\widetilde{0}}) = 0 \\
&\tfrac{\partial \rho_i}{\partial z^b} \neq 0 \neq \tfrac{\partial \rho_i}{\partial \overline{z^b}}
\end{aligned} \tag{1.1}$$

where the two real functions are assumed to be smooth with the indicated conditions. Notice that the two real functions do not mix the two coordinates. We essentially have two independent 1-dimensional totally real submanifolds of a corresponding complex manifold. Any holomorphic transformation ($z'^0 = f(z^0)$, $z'^{\widetilde{0}} = h(z^{\widetilde{0}})$) preserves the LCR-structure definition. Therefore it is called LCR-transformation. All these complex functions $(z^0(x), z^{\widetilde{0}}(x))$ will be generally called LCR-structure coordinates.

There is an important mathematical subtlety, which is going to play an essential role in the study of LCR-structure. In the general case of smooth LCR-manifolds, there is always a LCR-transformation in a neighborhood of every point p such that

$$\begin{aligned}
&\rho_1(\overline{z^0}, z^0) = \tfrac{z^0 - \overline{z^0}}{2i} - \phi(\tfrac{z^0 + \overline{z^0}}{2}) = 0 \quad , \quad \rho_2(\overline{z^{\widetilde{0}}}, z^{\widetilde{0}}) = \tfrac{z^{\widetilde{0}} - \overline{z^{\widetilde{0}}}}{2i} - \widetilde{\phi}(\tfrac{z^{\widetilde{0}} + \overline{z^{\widetilde{0}}}}{2}) = 0 \\
&\phi(0) = 0 = d\phi(0) \quad , \quad \widetilde{\phi}(0) = 0 = d\widetilde{\phi}(0)
\end{aligned} \tag{1.2}$$

where the two functions $\phi(\cdot)$ and $\widetilde{\phi}(\cdot)$ are smooth ($\equiv$ have all their derivatives in a neighborhood of the point p). If they are real analytic ($\equiv$ expand to a summable Taylor series in a neighborhood of the point p), there are[1] always independent analytic transformations giving them the simple forms

$$\begin{aligned}
&\rho_1(\overline{z^0}, z^0) = \tfrac{z^0 - \overline{z^0}}{2i} = 0 \quad , \quad \rho_2(\overline{z^{\widetilde{0}}}, z^{\widetilde{0}}) = \tfrac{z^{\widetilde{0}} - \overline{z^{\widetilde{0}}}}{2i} = 0 \\
&z^0 = u \quad , \quad z^{\widetilde{0}} = v
\end{aligned} \tag{1.3}$$

The two cotangent vectors (1-forms) of the LCR-manifold M are

$$\begin{aligned}
&\ell = \ell_\nu dx^\nu = i(\partial - \overline{\partial})\rho_1(\overline{z^0}, z^0) = i(\tfrac{dz^0 + d\overline{z^0}}{2i})|_M = du \\
&n = n_\nu dx^\nu = i(\partial - \overline{\partial})\rho_2(\overline{z^{\widetilde{0}}}, z^{\widetilde{0}}) = i(\tfrac{dz^{\widetilde{0}} + d\overline{z^{\widetilde{0}}}}{2i})|_M = dv \\
&\ell \wedge n \neq 0
\end{aligned} \tag{1.4}$$

If we multiply the defining real conditions with arbitrary non-vanishing functions $\Lambda(x)$ and $N(x)$, we find that the above 1-forms admit the symmetry

$$\ell' = \Lambda \ell \quad , \quad n' = N n \tag{1.5}$$

A basis $(\ell^\mu \partial_\mu, n^\mu \partial_\mu)$ of the tangent space of M is defined via the conditions

$$\begin{aligned} &(\ell^\mu \partial_\mu) \lrcorner (\ell_\nu dx^\nu) = 0 \quad , \quad (\ell^\mu \partial_\mu) \lrcorner (n_\nu dx^\nu) = 1 \\ &(n^\mu \partial_\mu) \lrcorner (\ell_\nu dx^\nu) = 1 \quad , \quad (n^\mu \partial_\mu) \lrcorner (n_\nu dx^\nu) = 0 \end{aligned} \tag{1.6}$$

The possibility to multiply the defining real conditions (1.1) with arbitrary non-vanishing functions $\Lambda(x)$ and $N(x)$, implies that the above two basis admit the symmetry

$$\begin{aligned} &\ell' = \Lambda \ell \quad , \quad n' = N n \\ &(\ell'^\mu \partial_\mu) = \tfrac{1}{N} (\ell^\mu \partial_\mu) \quad , \quad (n'^\mu \partial_\mu) = \tfrac{1}{\Lambda} (n^\mu \partial_\mu) \end{aligned} \tag{1.7}$$

which we will call dyad-Weyl symmetry.

Notice that we have not yet defined a metric. The LCR-structure does not need the notion of the metric. But this basis defines a 2-d symmetric tensor $g_{\mu\nu}$,

$$\begin{aligned} &[g_{\mu\nu}] \equiv \ell_\mu n_\nu + \ell_\nu n_\mu \\ &g \equiv \det(g_{\mu\nu}) = -[\det \begin{pmatrix} \ell_0 & \ell_1 \\ n_0 & n_1 \end{pmatrix}]^2 \end{aligned} \tag{1.8}$$

which will be used as a class of equivalent metrics, because of dyad-Weyl symmetry. These metrics have the characteristic property to make the two vectors of the basis null, i.e. $\ell^\mu \ell^\nu g_{\mu\nu} = 0 = n^\mu n^\nu g_{\mu\nu}$.

The well-known emergence of the algebraic structure in string theory appears if we consider the two structure coordinates z^0 and $z^{\tilde{0}}$ as two different points of a hypersurface (curve) of CP^2. An hypersurface of CP^2 is generally defined (in homogeneous coordinates)

$$\begin{aligned} &X^{mi} = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} \\ &K(X^{m1}) = K(X^{m2}) \end{aligned} \tag{1.9}$$

where $K(X^m)$ is the homogeneous polynomial (reducible or irreducible) that defines the Riemann surface. For the two points z^0 and $z^{\tilde{0}}$ to be different, the 3×2 matrix must have rank two. In this case they define a line of CP^2 which intersects the hypersurface at z^0 and $z^{\tilde{0}}$. The projective coordinates r_A ; $A = 0, 1$ of these lines are defined with the identity

$$\begin{aligned} &X^{mi} = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ -ir_A X^{A1} & -ir_A X^{A2} \end{pmatrix} \\ &\det \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \end{pmatrix} \neq 0 \end{aligned} \tag{1.10}$$

in the corresponding affine patch of the grassmannian projective manifold $G(3, 2)$. Assuming $z^0 = \frac{X^{11}}{X^{01}}$ and $z^{\tilde{0}} = \frac{X^{12}}{X^{02}}$, the structure coordinates are holomorphic functions of the projective coordinates r_A. And vice-versa, the projective coordinates are not always LCR-structure coordinates.

1.1 Examples of Riemann surfaces

The quadratic surfaces, which are usually called conics, are generally reduced to the (irreducible) homogeneous polynomial

$$\begin{aligned}
&K_2(Z^m) = (Z^0)^2 + (Z^1)^2 + (Z^2)^2 = 0 \\
&\partial_0 K_2 = 2(Z^0) = 0 \\
&\partial_1 K_2 = 2(Z^1) = 0 \\
&\partial_2 K_2 = 2(Z^2) = 0
\end{aligned} \tag{1.11}$$

Its tangent space is well defined, because it is well defined at all its points, i.e. $\partial_n K_2(Z^m) \neq 0$. Notice that

$$\begin{gathered}
\partial_n K_2(Z^m) = 0 \\
\Updownarrow \\
\partial_0 K_2 = 2(Z^0) = 0, \quad \partial_1 K_2 = 2(Z^1) = 0, \quad \partial_2 K_2 = 2(Z^2) = 0
\end{gathered} \tag{1.12}$$

admits only the solution $Z^m = 0$, $\forall m$, which is not an element of the projective space CP^2. Hence the conics are regular curves (without singular points). That is the tangent exists at all its points. It is a manifold.

The projective spaces are compact. They do not have "infinities". But their affine patches do have infinities. The infinity of the affine patch $Z^2 = 1$, is the subspace $Z^2 = 0$. In this patch the infinity of the above conic is found assuming the additional relation $Z^2 = 0$. The conic meets (intersects) the infinity (a line) at

$$\begin{aligned}
&K_2(Z^m) = (Z^0)^2 + (Z^1)^2 = 0 \\
&Z^0 : Z^1 = 1 : \pm i
\end{aligned} \tag{1.13}$$

which are two points (like its intersections with any other line).

We will now look at its tangents, using the method of the pencil of lines passing through a point X_0^m of the conic

$$\begin{aligned}
&X^m = X_0^m + sT^m \\
&K_2(X_0^m + sT^m) = \sum_m [(X_0^m)^2 + 2sX_0^m T^m + s^2 (T^m)^2] = \\
&\quad = 2s \sum_m [X_0^m T^m] + s^2 \sum_m [(T^m)^2] = 0
\end{aligned} \tag{1.14}$$

where T^m are the slopes of the line and we sum over the repeated index m. The intersection index $I(X_0^m, K_2, L)$ of this line L is the number of solutions of $K_2(X_0^m + sT^m)$ relative to s. Bezout's theorem tell us that now, there are generally two solutions, one of them being $s = 0$, which corresponds to the point X_0^m of the conic. If T^m satisfies the relation $X_0^m T^m = 0$ i.e. if the line is tangent at X_0^m, then the solution $s = 0$ is double. So the tangents of regular points are found.

In order to find and understand the parametrization and multi-sheet algebraic surface, we have to consider an affine patch. In the affine patch $Z^0 = 1$ and the notation $x = -i\frac{Z^1}{Z^0}$, $y = -i\frac{Z^2}{Z^0}$ we have the complex circle

$$K_2(x,y) = 1 - x^2 - y^2 \tag{1.15}$$

All its finite points are regular because

$$\begin{aligned} &K_2(x,y) = 1 - x^2 - y^2 = 0 \\ &\partial_x K_2 = -2x = 0 \\ &\partial_y K_2 = -2y = 0 \end{aligned} \tag{1.16}$$

do not have a (complex) solution.

Starting from a regular solution and using its pencil of lines, we can generally find the other points where each line intersects the surface. In the present case, I consider the point $(x = 1, y = 0)$. Then the pencil of lines

$$\begin{aligned} &\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ t \end{pmatrix} \\ &y = t(x-1) \end{aligned} \tag{1.17}$$

imply the equation

$$\begin{aligned} &(1-x)[x(t^2+1) - (t^2-1)] = 0 \\ &x = \tfrac{(t^2-1)}{(t^2+1)} \quad , \quad y = \tfrac{-2t}{(t^2+1)} \end{aligned} \tag{1.18}$$

which gives the second intersection point for each line. Returning back to homogeneous coordinates we find the rational representation

$$\begin{aligned} &x = -i\tfrac{Z^1}{Z^0} = \tfrac{(t^2-1)}{(t^2+1)} \quad , \quad y = -i\tfrac{Z^2}{Z^0} = \tfrac{-2t}{(t^2+1)} \\ &Z^0 : Z^1 : Z^2 = (t^2+1) : i(t^2-1) : -2it \end{aligned} \tag{1.19}$$

The infinity $t = \infty$ is covered by simply making the transformation $t = \frac{1}{t'}$.

The double-valuedness of the quadratic curve is simply indicated by its degree two. It is also indicated by the highest degree of the polynomials which appear in the rational representation. Notice that in the conics the two values $t = \pm c \neq 0$ have one x and two values $\pm\frac{-2c}{(c^2+1)}$

$$\begin{aligned} &t = \pm c \neq 0 \quad \rightarrow \quad (x,y) = (\tfrac{(c^2-1)}{(c^2+1)}, \pm\tfrac{-2c}{(c^2+1)}) \\ &t = 0 \quad \rightarrow \quad (x,y) = (-1,0) \\ &t = \infty \quad \rightarrow \quad (x,y) = (1,0) \end{aligned} \tag{1.20}$$

where $t = 0, \infty$ are the branch points, which have one value. But these points have nothing special. They become special by the chosen (by us) coordinate system! A rotation of the coordinate system changes the branch points! A general projective transformation preserves the characteristics of an algebraic projective curve, but it changes the role of each point.

Let us now consider the cubic curve

$$K_3(Z^m) = (Z^0)(Z^2)^2 + \lambda(Z^0)(Z^1)^2 + (Z^1)^3$$

$$\begin{aligned} \partial_0 K_3 &= (Z^2)^2 + \lambda(Z^1)^2 = 0 \\ \partial_1 K_3 &= 2\lambda(Z^0)(Z^1) + 3(Z^1)^2 = 0 \\ \partial_2 K_3 &= 2(Z^0)(Z^2) = 0 \end{aligned} \tag{1.21}$$

It has the singular point

$$Z^m = 1 : 0 : 0 \tag{1.22}$$

We will now look at its tangents, using the method of the pencil of lines passing through this point $X_0^m = 1 : 0 : 0$ of the cubic

$$\begin{aligned} & X^m = X_0^m + sT^m \\ & K_3(X_0^m + sT^m) = s^2[(T^2)^2 + \lambda(T^1)^2] + \\ & \quad + s^3[T^0(T^2)^2 + T^0(T^1)^2 + (T^1)^3] \end{aligned} \tag{1.23}$$

where T^m are the slopes of the line. Hence this point is a double point, because the lowest degree of s is two. For $\lambda \neq 0$ there are two different tangents $(T^1 : T^2) = (1 : \pm i\lambda)$. Hence the point $(0,0)$ is a *node*. For $\lambda = 0$ we have one solution, which is a *cusp*. This manifold is singular at this point, because its corresponding tangent space has dimension lower than the manifold itself.

The tangent lines at this point in the affine patch $Z^0 = 1$ are

$$\begin{aligned} & x = \tfrac{Z^1}{Z^0} \quad , \quad y = \tfrac{Z^2}{Z^0} \\ & K_3(x,y) = \lambda x^2 + x^3 + y^2 \\ & K_3(0 + sX, 0 + sY) = s^2(\lambda X^2 + Y^2) + s^3 X^3 \end{aligned} \tag{1.24}$$

For $\lambda \neq 0$ there are two different tangents $(1, \pm i\sqrt{\lambda})$. Hence the point $(0,0)$ is a *node*. For $\lambda = 0$ we have the previous *cusp*.

The parametrization of this curve is found with the pencil of lines

$$\begin{aligned} & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ t \end{pmatrix} \\ & y = tx \end{aligned} \tag{1.25}$$

but now we solve relative to the slope t. We find

$$\begin{aligned} & K_3(x,y) = x^2(\lambda + x + t^2) = 0 \\ & x = -(\lambda + t^2) = \tfrac{Z^1}{Z^0} \quad , \quad y = -t(\lambda + t^2) = \tfrac{Z^2}{Z^0} \\ & Z^m = [1 : -(\lambda + t^2) : -t(\lambda + t^2)] \end{aligned} \tag{1.26}$$

The multiple points do not cause any problem, if they admit the same number of different tangents, because the Riemann surface can be generated by analytically extending one-value $y_k(x)$. But if the two or more tangents at a multiple point coincide, we cannot make the extension! This will appear as a branch point. The singular points are "regularized" using blow-ups. This is an algebraic technique to consider them as points of a larger projective space.

2 THE 2-D PSEUDO-CONFORMAL LAGRANGIAN

The general framework of the two dimensional (2-d) pseudo-conformal field theory (PCFT) will be studied with the simple 2-d conformal action

$$\begin{array}{l} I_G = \int_M \ell^\mu(\partial_\mu\Psi_j)n^\nu(\partial_\nu\Psi_k)\theta_{jk}\sqrt{-g}d^2x \\ g_{\mu\nu} \equiv \ell_\mu n_\nu + \ell_\nu n_\mu \quad , \quad g = -[\det\begin{pmatrix} \ell_0 & \ell_1 \\ n_0 & n_1 \end{pmatrix}]^2 \end{array} \tag{2.1}$$

where $\Psi_j(x)$ is a field defined on the (differentiable) manifold M , which takes values in a vector space with internal metric θ_{jk} generally assumed diagonal with ± 1 elements. A basis $(\ell^\mu\partial_\mu, n^\mu\partial_\mu)$ of the tangent space of M is also used. Its dual basis of the cotangent space $(\ell_\nu dx^\nu, n_\nu dx^\nu)$ is defined via the conditions

$$\begin{array}{l} (\ell^\mu\partial_\mu)\lrcorner(\ell_\nu dx^\nu) = 0 \quad , \quad (\ell^\mu\partial_\mu)\lrcorner(n_\nu dx^\nu) = 1 \\ \\ (n^\mu\partial_\mu)\lrcorner(\ell_\nu dx^\nu) = 1 \quad , \quad (n^\mu\partial_\mu)\lrcorner(n_\nu dx^\nu) = 0 \end{array} \tag{2.2}$$

This basis defines a 2-d symmetric tensor $g_{\mu\nu}$, which we simply notationally use in order to proper define the integral.

The field equations are usually derived and found to be

$$\begin{array}{l} \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Psi_j) = 0 \\ \\ \sum\limits_{jk}(\ell^\mu\partial_\mu\Psi_j)(\ell^\mu\partial_\mu\Psi_k)\theta_{jk} = 0 \quad , \quad \sum\limits_{j}(n^\mu\partial_\mu\Psi_j)(n^\mu\partial_\mu\Psi_k)\theta_{jk} = 0 \end{array} \tag{2.3}$$

Notice that if the constant matrix θ_{jk} is positive definite we have only the trivial solution $\Psi_j = const$. If the manifold has a boundary, additional relations for $\Psi_j(x)$ at the boundary are yielded.

The action is the Polyakov action and it is compatible with the 2-dimensional LCR-structure, because it is invariant under the local dyad-Weyl transformation

$$(\ell'_\nu dx^\nu, n'_\nu dx^\nu) = (\Lambda\ell_\nu dx^\nu, Nn_\nu dx^\nu) \quad , \quad \Lambda(x) \neq 0 \neq N(x) \tag{2.4}$$

Notice that the dyad-Weyl transformation is equivalent to the ordinary Weyl transformation

$$g'_{\mu\nu} = \Lambda N g_{\mu\nu} \tag{2.5}$$

which will not be the case for action of the 4-dimensional PCFT.

A 2-dimensional manifold locally admits a real coordinate system (t, x) such that

$$
\begin{gathered}
\ell_\nu dx^\nu = \Lambda du = \Lambda d(t-x) \quad , \quad n_\nu dx^\nu = N dv = N d(t+x) \\
g_{\mu\nu} dx^\mu dx^\nu = 2\Lambda N (du)(dv) = 2\Lambda N((dt)^2 - (dx)^2)
\end{gathered}
\tag{2.6}
$$

Notice that in this LCR coordinate system the action is metric independent

$$
\begin{aligned}
I_G = & \int_M \ell^\mu (\partial_\mu \Psi_j) n^\nu (\partial_\nu \Psi_k) \theta_{jk} \ell \wedge n = \\
= & \int_M (\partial_v \Psi_j)(\partial_u \Psi_k) \theta_{jk} du dv
\end{aligned}
\tag{2.7}
$$

while it is not a topological action.

Hence the question is raised whether we can formulate a 4-dimensional LCR-structure and write down a compatible action which is metric independent without being topological. I found that it can be done, but we need a better knowledge of the Cartan moving frame formalism, which I have to describe below. I will start first with the Newman-Penrose Cartan formulation of general relativity and after I will proceed to the 4-dimensional lorentzian CR-structure in order to make clear that the riemannian structure is different from the LCR-structure.

3 CARTAN FORMULATION OF GENERAL RELATIVITY

The fundamental property of the 2-dimensional Polyakov action (2.7) is its metric independence in the coordinate system of structure coordinates, where the metric takes the off-diagonal form $ds^2 = \Lambda N du dv$. Flaherty has observed[12],[13], that if a 4-dimensional metric admits a geodetic and shear-free null tetrad $(\ell_\mu, m_\mu, n_\mu, \overline{m}_\mu)$, there is a generally complex coordinate system $(z^\alpha, z^{\widetilde{\beta}})$: $\alpha, \beta = 0, 1$, such that the 4-dimensional metric takes the off-diagonal form

$$
\begin{gathered}
ds^2 = g_{\alpha\widetilde{\beta}} dz^\alpha dz^{\widetilde{\beta}} \\
g_{\mu\nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu
\end{gathered}
\tag{3.1}
$$

The moving frames of E. Cartan[20] is the convenient framework to study these notions and the LCR-structure formalism. In fact the Newman-Penrose formalism[6] in general relativity is just the Cartan formalism adapted to a null tetrad. Therefore we will describe Einstein's gravity in this formalism in order to better understand its relation with the LCR-structure.

Every coordinate system x^μ defines a frame ∂_μ in the tangent space and dx^μ in the cotangent space, which satisfy the following relations

$$
\begin{gathered}
[\partial_\mu,\ \partial_\nu] = 0 \quad , \quad \partial_\mu \lrcorner dx^\nu = \delta^\nu_\mu \\
\partial_\mu \lrcorner (dx^\nu \wedge dx^\rho) = \delta^\nu_\mu dx^\rho - \delta^\rho_\mu dx^\nu
\end{gathered}
\tag{3.2}
$$

A moving frame $e^\mu_a \partial_\mu$ of the tangent space of a smooth manifold (spacetime) satisfies the general commutation relations

$$
\begin{aligned}
&[(e_a^\mu \partial_\mu)\ ,\ (e_b^\nu \partial_\nu)] = c_{ab}{}^{d}\ (e_d^\rho \partial_\rho) \\
&c_{ab}{}^{d} + c_{ba}{}^{d} = 0
\end{aligned}
\tag{3.3}
$$

$c_{ab}{}^{d}$ generally depend on x. If $c_{ab}{}^{d}$ are constants, the manifold has a group structure and the corresponding vectors of the moving frame of the group manifold correspond to the generators of its Lie algebra. Its dual frame in the cotangent space is $e^b := e_\nu^b dx^\nu$, defined by the relations

$$
(e_a^\mu \partial_\mu) \lrcorner (e_\nu^b dx^\nu) = e_a^\mu e_\nu^b (\partial_\mu \lrcorner dx^\nu) = e_a^\mu e_\nu^b \delta_\mu^\nu = e_a^\mu e_\mu^b = \delta_a^b \tag{3.4}
$$

Then, they satisfy the relations

$$
\begin{aligned}
&de^a + \omega^a{}_b \wedge e^b = 0 \\
&\omega_{ab} = \eta_{ac} \omega^c{}_b = -\omega_{ba} \\
&c_{ab}{}^{d} = \omega^d{}_{b\mu} e_a^\mu - \omega^d{}_{a\mu} e_b^\mu
\end{aligned}
\tag{3.5}
$$

The curvature of the manifold and the Bianchi identities are

$$
\begin{aligned}
&\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \\
&d\Omega^a{}_b + \Omega^a{}_c \wedge \Omega^c{}_b = 0
\end{aligned}
\tag{3.6}
$$

In the context of Cartan, the curvature measures how much the manifold differs from a group (of the same dimension) after its osculation with the precise group.

The Newman-Penrose (NP-) formalism makes the following identifications

$$
\begin{aligned}
&e_\mu^0 dx^\mu =: \ell_\mu dx^\mu\ ,\ e_\mu^1 dx^\mu =: m_\mu dx^\mu\ ,\ e_\mu^{\widetilde{0}} dx^\mu =: n_\mu dx^\mu\ ,\ e_\mu^{\widetilde{1}} dx^\mu =: \overline{m}_\mu dx^\mu \\
&e_0^\mu \partial_\mu =: n^\mu \partial_\mu\ ,\ e_1^\mu \partial_\mu =: -\overline{m}^\mu \partial_\mu\ ,\ e_{\widetilde{0}}^\mu \partial_\mu =: \ell^\mu \partial_\mu\ ,\ e_{\widetilde{1}}^\mu \partial_\mu =: -m^\mu \partial_\mu \\
&e_a^\mu e_\mu^b = \delta_a^b\ \ ,\ \ e_a^\mu e_\nu^a = \delta_\nu^\mu
\end{aligned}
\tag{3.7}
$$

and the metric has the form

$$
\begin{aligned}
&g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b\ \ ,\ \ g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu \\
&\eta_{ab} = \eta^{ab} = \begin{pmatrix} 0 & \eta_{\alpha\widetilde{\beta}} \\ \eta_{\widetilde{\alpha}\beta} & 0 \end{pmatrix}\ \ ,\ \eta_{\alpha\widetilde{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
\tag{3.8}
$$

The first latin indices $a, b, ...$ have been accommodated to the LCR-structure definition, which will be presented in the next section. They take the values $(0, 1; \widetilde{0}, \widetilde{1})$.

In the Cartan-NP-formalism, the Cartan connection is denoted with a large number of symbols, which turn out to be very useful. We precisely have the following definition[6] of the NP-coefficients

$$
\begin{aligned}
&[\ell^\mu\partial_\mu\ ,\ n^\nu\partial_\nu]=-(\gamma+\overline{\gamma})\ell^\rho\partial_\rho-(\varepsilon+\overline{\varepsilon})n^\rho\partial_\rho+(\overline{\tau}+\pi)m^\rho\partial_\rho+(\tau+\overline{\pi})\overline{m}^\rho\partial_\rho\\
&[\ell^\mu\partial_\mu\ ,\ m^\nu\partial_\nu]=(\overline{\pi}-\overline{\alpha}-\beta)\ell^\rho\partial_\rho-\kappa n^\rho\partial_\rho+(\overline{\rho}+\varepsilon-\overline{\varepsilon})m^\rho\partial_\rho+\sigma\overline{m}^\rho\partial_\rho\\
&[n^\mu\partial_\mu\ ,\ m^\nu\partial_\nu]=\overline{\nu}\ell^\rho\partial_\rho+(\overline{\alpha}+\beta-\tau)n^\rho\partial_\rho+(\gamma-\overline{\gamma}-\mu)m^\rho\partial_\rho-\overline{\lambda}\overline{m}^\rho\partial_\rho\\
&[m^\mu\partial_\mu\ ,\ \overline{m}^\nu\partial_\nu]=(\mu-\overline{\mu})\ell^\rho\partial_\rho+(\rho-\overline{\rho})n^\rho\partial_\rho+(\overline{\beta}-\alpha)m^\rho\partial_\rho+(\overline{\alpha}-\beta)\overline{m}^\rho\partial_\rho
\end{aligned}
\tag{3.9}
$$

which is equivalent to

$$
\begin{aligned}
d\ell=&-(\varepsilon+\overline{\varepsilon})\ell\wedge n+(\alpha+\overline{\beta}-\overline{\tau})\ell\wedge m+(\overline{\alpha}+\beta-\tau)\ell\wedge\overline{m}-\\
&-\overline{\kappa}n\wedge m-\kappa n\wedge\overline{m}+(\rho-\overline{\rho})m\wedge\overline{m}\\
dn=&-(\gamma+\overline{\gamma})\ell\wedge n+\nu\ell\wedge m+\overline{\nu}\ell\wedge\overline{m}+(\pi-\alpha-\overline{\beta})n\wedge m+\\
&+(\overline{\pi}-\overline{\alpha}-\beta)n\wedge\overline{m}+(\mu-\overline{\mu})m\wedge\overline{m}\\
dm=&-(\tau+\overline{\pi})\ell\wedge n+(\gamma-\overline{\gamma}+\overline{\mu})\ell\wedge m+\overline{\lambda}\ell\wedge\overline{m}+\\
&+(\varepsilon-\overline{\varepsilon}-\rho)n\wedge m-\sigma n\wedge\overline{m}+(\beta-\overline{\alpha})m\wedge\overline{m}
\end{aligned}
\tag{3.10}
$$

The following direct definitions of the NP-coefficients will be computationally very useful

$$
\boxed{
\begin{array}{|l|}
\hline
\alpha=\frac{1}{4}[(\ell n\partial\overline{m})+(\ell\overline{m}\partial n)-(n\overline{m}\partial\ell)-2(m\overline{m}\partial\overline{m})]\\ \hline
\beta=\frac{1}{4}[(\ell n\partial m)+(\ell m\partial n)-(nm\partial\ell)-2(m\overline{m}\partial m)]\\ \hline
\gamma=\frac{1}{4}[(nm\partial\overline{m})-(n\overline{m}\partial m)-(m\overline{m}\partial n)+2(\ell n\partial n)]\\ \hline
\varepsilon=\frac{1}{4}[(\ell m\partial\overline{m})-(\ell\overline{m}\partial m)-(m\overline{m}\partial\ell)+2(\ell n\partial\ell)]\\ \hline
\mu=-\frac{1}{2}[(m\overline{m}\partial n)+(nm\partial\overline{m})+(n\overline{m}\partial m)]\\ \hline
\pi=\frac{1}{2}[(\ell n\partial\overline{m})-(n\overline{m}\partial\ell)-(\ell\overline{m}\partial n)]\\ \hline
\rho=\frac{1}{2}[(\ell\overline{m}\partial m)+(\ell m\partial\overline{m})-(m\overline{m}\partial\ell)]\\ \hline
\tau=\frac{1}{2}[(nm\partial\ell)+(\ell m\partial n)+(\ell n\partial m)]\\ \hline
\kappa=(\ell m\partial\ell)\quad,\quad\sigma=(\ell m\partial m)\\ \hline
\nu=-(n\overline{m}\partial n)\quad,\quad\lambda=-(n\overline{m}\partial\overline{m})\\ \hline
\end{array}}
\tag{3.11}
$$

where the symbols (...) are constructed according to the rule of the following example $(\ell m\partial n):=(\ell^\mu m^\nu-\ell^\nu m^\mu)(\partial_\mu n_\nu)$. In the NP-formalism the ten real quantities of the Weyl tensor $C_{\mu\nu\rho\sigma}$ are represented with the following five complex scalars

$$
\begin{aligned}
&\Psi_0=-C_{\mu\nu\rho\sigma}\ell^\mu m^\nu\ell^\rho m^\sigma\quad,\quad\Psi_1=-C_{\mu\nu\rho\sigma}\ell^\mu n^\nu\ell^\rho m^\sigma\\
&\Psi_2=-C_{\mu\nu\rho\sigma}\ell^\mu m^\nu\overline{m}^\rho n^\sigma\quad,\quad\Psi_3=-C_{\mu\nu\rho\sigma}\ell^\mu n^\nu\overline{m}^\rho n^\sigma\\
&\Psi_4=-C_{\mu\nu\rho\sigma}n^\mu\overline{m}^\nu n^\rho\overline{m}^\sigma
\end{aligned}
\tag{3.12}
$$

In general relativity, the fundamental quantity is the metric tensor. Therefore the use of a moving frame generates a local $SO(1,3)$ symmetry of moving frames. In the NP-formalism this symmetry is fragmented (as everything) into the following subgroups

$$
\begin{aligned}
&(I):\ \textit{rotations which leave } \ell \textit{ invariant}\\
&(II):\ \textit{rotations which leave } n \textit{ invariant}\\
&(III):\ \textit{rotations in the } (m,\overline{m})-\textit{plane},\\
&\qquad \textit{which leave } \ell \textit{ and } n \textit{ invariant}
\end{aligned}
\tag{3.13}
$$

These rotations imply the following transformations of the null tetrad

$$\begin{array}{l} (I): \ \ell' = \ell, \ n' = n + \overline{a}m + a\overline{m} + a\overline{a}\ell, \ m' = m + a\ell \\ (II): \ n' = n, \ \ell' = l + \overline{b}m + b\overline{m} + b\overline{b}n, \ m' = m + bn \\ (III): \ \ell' = A^{-1}\ell, \ n' = An, \ m' = e^{i\varphi}m \end{array} \tag{3.14}$$

where a, b are complex functions, A is a real function and φ is a real angle function.

The NP-formalism has a very large number of symbols, but it is really very effective, because it singles out the geodetic and shear-free properties of the null tetrad. The geodetic and shear-free null tetrad satisfies the relations $\kappa = \sigma = 0 = \nu = \lambda$, which are the fundamental equations of the 4-dimensional LCR-structure. Notice that the null tetrad, which satisfies these conditions, has the basic property of the Frobenius theorem

$$\begin{array}{l} [\ell^\mu \partial_\mu \ , \ m^\nu \partial_\nu] = (\overline{\pi} - \overline{\alpha} - \beta)\ell^\rho \partial_\rho + (\overline{\rho} + \varepsilon - \overline{\varepsilon})m^\rho \partial_\rho \\ [n^\mu \partial_\mu \ , \ \overline{m}^\nu \partial_\nu] = (\alpha + \overline{\beta} - \overline{\tau})n^\rho \partial_\rho + (\overline{\gamma} - \gamma - \overline{\mu})\overline{m}^\rho \partial_\rho \end{array} \tag{3.15}$$

The corresponding cotangent basis satisfies the equivalent relations

$$\begin{array}{l} d\ell = [(\varepsilon + \overline{\varepsilon})n - (\alpha + \overline{\beta} - \overline{\tau})m - (\overline{\alpha} + \beta - \tau)\overline{m}] \wedge \ell + (\rho - \overline{\rho})m \wedge \overline{m} \\ dm = [(\gamma - \overline{\gamma} + \overline{\mu})\ell + (\varepsilon - \overline{\varepsilon} - \rho)n + (\overline{\alpha} - \beta)\overline{m}] \wedge m - (\tau + \overline{\pi})\ell \wedge n \\ \\ dn = [-(\gamma + \overline{\gamma})\ell + (\alpha + \overline{\beta} - \pi)m + (\overline{\alpha} + \beta - \overline{\pi})\overline{m}] \wedge n + (\mu - \overline{\mu})m \wedge \overline{m} \\ d\overline{m} = [(\overline{\gamma} - \gamma + \mu)\ell + (\overline{\varepsilon} - \varepsilon - \overline{\rho})n + (\alpha - \overline{\beta})m] \wedge \overline{m} - (\overline{\tau} + \pi)\ell \wedge n \end{array} \tag{3.16}$$

These conditions permit the application of the holomorphic Frobenius theorem. For that we trivially complexify the coordinates x^μ and apply the theorem to the two independent pairs (ℓ, m) and $(n, \widetilde{m})$, viewed as holomorphic vector fields on the ambient complex manifold. The holomorphic Frobenius theorem states that there are four independent complex functions $(z^\alpha, \ z^{\widetilde{\alpha}})$, $\alpha = 0, \ 1$, such that

$$\begin{array}{l} dz^\alpha = f_0^\alpha \ \ell_\mu dx^\mu + f_1^\alpha \ m_\mu dx^\mu \quad , \quad dz^{\widetilde{\alpha}} = f_{\widetilde{0}}^{\widetilde{\alpha}} \ n_\mu dx^\mu + f_{\widetilde{1}}^{\widetilde{\alpha}} \ \widetilde{m}_\mu dx^\mu \\ \\ \ell = \ell_\alpha dz^\alpha \ , \ m = m_\alpha dz^\alpha \quad ; \quad n = n_{\widetilde{\alpha}} dz^{\widetilde{\alpha}} \ , \ \widetilde{m} = \widetilde{m}_{\widetilde{\alpha}} dz^{\widetilde{\alpha}} \end{array} \tag{3.17}$$

When we return back to the real spacetime the metric takes the off-diagonal form

$$g_{\mu\nu} dx^\mu dx^\nu = 2(\ell_\alpha n_{\widetilde{\beta}} - m_\alpha \overline{m}_{\widetilde{\beta}}) dz^\alpha dz^{\widetilde{\beta}} \tag{3.18}$$

Recall that it is exactly this property, that we are looking to transfer to four dimensions.

Einstein's idea was to consider the lorentzian metric $g_{\mu\nu}$ as the fundamental quantity of gravitation and the Newton potential and equations of motion are derived. The basic principle of pseudo-conformal field theory is to consider the

LCR-tetrad as the fundamental quantity from which electromagnetism, gravitation and weak interactions are derived. The particles are the wavefront singularities of the distributional solutions, which appear when we return back from the ambient complex manifold down to the real spacetime.

I think it is time to make some fundamental differences between the Cartan formalism (based on moving frames) than the metric of riemannian geometry. The moving frame is generally assumed linear independent. It implies the riemannian geometry by assuming it orthonormal. In order to measure a length, we have to impose the precise symmetric matrix η_{ab} that defines (through the tetrad) the metric. The local Lorentz group $SO(1,3)$ of Cartan is implied by the matrix η_{ab} and it is valid for all spacetimes, flat or curved either. The LCR-structure by its definition $\kappa = \sigma = 0 = \nu = \lambda$, breaks the Cartan local symmetry $SO(1,3)$, but it respects the general diffeomorphism group.

3.1 Goldberg-Sachs and Kerr's theorems

The Goldberg-Sachs theorem states that if ℓ is geodetic and shear-free null vector, the corresponding Weyl scalar Ψ_0 vanishes. Notice that the inverse is not true. We will apply this theorem for metrics which admit two geodetic and shear-free null congruences. That is, a geodetic and shear-free null tetrad. Hence starting from a general regular tetrad with non-vanishing Weyl scalars, we can make class I and II rotations such that

$$\begin{array}{l} \Psi_0' = \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0 \\ \Psi_4' = \Psi_4 + 4\overline{a}\Psi_3 + 6\overline{a}^2\Psi_2 + 4\overline{a}^3\Psi_1 + \overline{a}^4\Psi_0 = 0 \end{array} \tag{3.19}$$

Notice that these two equations are projectively equivalent, because if the first has a solution b, then $\frac{1}{b}$ is solution of the second one. Hence at every point, the maximum number of roots can be four. This implies a limit of geodetic and shear-free null tetrads, which are compatible with a metric. We will use this limited number of projective solutions to restrict the number of the fermionic elementary particle generations (families).

If the metric is (conformally) flat, no restriction on the number of compatible LCR-structures is implied but the Kerr theorem permits an algebraic computation of these LCR-structures. Let us consider the trivial null tetrad of the Minkowski metric

$$\begin{array}{l} \ell = \partial_v \quad , \quad n = \partial_u \quad , \quad m = -\partial_{\overline{\zeta}} \\ \\ u = t - z \quad , \quad u = t + z \quad , \quad \zeta = x + iy \end{array} \tag{3.20}$$

All its NP spin coefficients vanish. After a class I and II rotations (3.14) we find the PDEs

$$\begin{array}{l} \partial_{\overline{\zeta}} a - a\partial_v a = 0 \quad , \quad \partial_u a - a\partial_\zeta a = 0 \\ \\ \partial_{\overline{\zeta}} b - b\partial_u b = 0 \quad , \quad \partial_v b - b\partial_\zeta b = 0 \end{array} \tag{3.21}$$

The two pairs of PDEs are similar, therefore we will describe Kerr's solution for the first pair and apply it to the second one too. The first pair may be viewed as the integrability problem of the following PDEs

$$(\partial_{\overline{\zeta}} - a\partial_v)X = 0 \quad , \quad (\partial_u - a\partial_\zeta)X = 0 \tag{3.22}$$

which admits solutions if

$$[(\partial_{\overline{\zeta}} - a\partial_v), (\partial_u - a\partial_\zeta)] = A(\partial_{\overline{\zeta}} - a\partial_v) + B(\partial_u - a\partial_\zeta) \tag{3.23}$$

A straightforward calculation implies that it is valid if a satisfies the initial PDEs (3.21). For such an a the system (3.22) admits the two solutions $X_1 = v + a\overline{\zeta}$ and $X_2 = \zeta + au$. Hence a satisfies the initial PDEs if it is a root of a general analytic function

$$K_1(a, v + a\overline{\zeta}, \zeta + au) = 0 \tag{3.24}$$

which I will generally call Kerr function. By complete analogy, the general solution of the second pair is a root of an analytic function

$$K_2(b, u + b\overline{\zeta}, \zeta + bv) = 0 \tag{3.25}$$

Penrose saw the Kerr function as defining a hypersurface of $CP(3)$ and introduced his twistor program. PCFT needs two geodetic and shear-free null congruences, therefore we will need two points lying in a reducible or irreducible hypersurface of $CP(3)$. The first point will provide a solution of the first pair and the second point of the second pair of the four PDEs (3.14). Besides in PCFT the emergence of a surface of $CP(3)$ occurs even in the cased of curved LCR-manifolds.

3.2 Spinorial formalism of general relativity

The indices of a 2-dimensional spinor $\lambda^A; A = 0, 1$ are lowered and raised as follows[28]

$$\begin{aligned}
&\lambda^A = \epsilon^{AB}\lambda_B \quad , \quad \lambda_C = \lambda^B \epsilon_{BC} \\
&\lambda^A \xi_A = \lambda^A \xi^B \epsilon_{BA} = -\lambda_B \xi^B \quad , \quad \lambda^A \lambda_A = 0 \\
&\epsilon^{AB} = \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \epsilon_A^B = \epsilon_{AC}\epsilon^{BC} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{3.26}$$

A spinor dyad is a basis of two dimensional spinors $\lambda^{Aj}; A = 0, 1; j = 1, 2$ such that

$$\lambda_A^1 \lambda^{A2} \equiv \lambda^{B1}\lambda^{A2}\epsilon_{BA} = 1 = -\lambda_A^2 \lambda^{A1} \tag{3.27}$$

One can find that this normalization is invariant under the transformation with any element S of the unimodular group $SL(2, \mathbb{C})$, which is implied by

$$\lambda'^{B1}\lambda'^{A2}\epsilon_{BA} = S^B_{\ C}\lambda^{C1}S^A_{\ D}\lambda^{D2}\epsilon_{BA} = \lambda^{C1}\lambda^{D2}\epsilon_{CD}, \quad \forall\lambda^{B1},\lambda^{A2}$$
$$S^B_{\ C}S^A_{\ D}\epsilon_{BA} = \epsilon_{CD} \tag{3.28}$$

In two dimensions $SL(2,\mathbb{C})$ has the two non-equivalent representations $S^A_{\ D}$ and $\overline{S^A_{\ D}} = \overline{S}^{A'}_{\ D'}$

Because of the homomorphism between $SL(2,\mathbb{C})$ and the (orthochronous) Lorentz group, we may use the 4-dimensional basis of 2×2 hermitian matrices

$$\begin{array}{l} \sigma^0_{A'B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \sigma^1_{A'B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2_{A'B} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma^3_{A'B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \\ \eta^{\mu\nu}\ \sigma^{A'A}_\mu\ \sigma^{B'B}_\nu = 2\epsilon^{A'B'}\epsilon^{AB} \quad , \quad \eta_{\mu\nu}\sigma^\mu_{A'A}\sigma^\nu_{B'B} = 2\epsilon_{A'B'}\epsilon_{AB} \\ \epsilon_{A'B'}\epsilon_{AB}\ \sigma^{A'A}_\mu\ \sigma^{B'B}_\nu = 2\eta_{\mu\nu} \end{array} \tag{3.29}$$

to construct a vector field $\xi^a = \sigma^a_{A'A}\overline{\lambda}^{A'1}\lambda^{A2}$. This can be extended to the curved spacetime by simply considering $\xi^\mu = e^\mu_a\sigma^a_{A'A}\overline{\lambda}^{A'1}\lambda^{A2}$. The spinor formalism is very useful, because any null vector k^μ takes the

$$k^\mu = e^\mu_a\sigma^a_{A'A}\overline{\lambda}^{A'}\lambda^A \quad , \quad k^\mu k^\nu g_{\mu\nu} = 0 \tag{3.30}$$

and a null tetrad has the form

$$\ell^\mu = \tfrac{1}{\sqrt{2}}e_a^{\ \mu}\sigma^a_{A'A}\overline{\lambda}^{A'1}\lambda^{A1}, \quad n^\mu = \tfrac{1}{\sqrt{2}}e_a^{\ \mu}\sigma^a_{A'A}\overline{\lambda}^{A'2}\lambda^{A2}, \quad m^\mu = \tfrac{1}{\sqrt{2}}e_a^{\ \mu}\sigma^a_{A'A}\overline{\lambda}^{A'2}\lambda^{A1} \tag{3.31}$$

relative to a spinor dyad λ^{Aj}. These forms permit[28] to write all the tensors of general relativity with world indices into formally equivalent tensors with spinorial indices.

4 DEFINITION OF 4-D LORENTZIAN CR-STRUCTURE

The four dimensional lorentzian CR-structure is defined as a frame with two real and one complex vector fields $(\ell^\mu\partial_\mu, m^\mu\partial_\mu; n^\mu\partial_\mu, \overline{m}^\mu\partial_\mu)$ on a smooth four dimensional manifold, which satisfy the commutation relations

$$\begin{array}{l} [\ell^\mu\partial_\mu\ ,\ m^\nu\partial_\nu] = h^{\widetilde{0}}_{\widetilde{0}}\ell^\rho\partial_\rho + h^{\widetilde{1}}_{\widetilde{0}}m^\rho\partial_\rho \\ [n^\mu\partial_\mu\ ,\ \overline{m}^\nu\partial_\nu] = h^0_0 n^\rho\partial_\rho + h^1_0\overline{m}^\rho\partial_\rho \end{array} \tag{4.1}$$

In the terminology of integrable systems, they are usually called Lax pairs. This definition is equivalent with the existence of a coframe $(\ell'_\mu dx^\mu, m'_\mu dx^\mu; n'_\mu dx^\mu, \overline{m'}_\mu dx^\mu)$

of two real and one complex 1-forms determined via the non-vanishing duality relations

$$
\begin{aligned}
&(\ell^{\mu}\partial_{\mu})\lrcorner(n'_{\nu}dx^{\nu}) = (n^{\mu}\partial_{\mu})\lrcorner(\ell'_{\nu}dx^{\nu}) = 1 \\
&(m^{\mu}\partial_{\mu})\lrcorner(\overline{m'}_{\nu}dx^{\nu}) = (\overline{m}^{\mu}\partial_{\mu})\lrcorner(m'_{\nu}dx^{\nu}) = -1 \\
\\
&(\ell'_{\nu}dx^{\nu}) \wedge (m'_{\nu}dx^{\nu}) \wedge (n'_{\nu}dx^{\nu}) \wedge (\overline{m'}_{\nu}dx^{\nu}) \neq 0
\end{aligned}
\tag{4.2}
$$

The other contractions vanish. The implied relations are

$$
\begin{aligned}
&(\ell^{\mu}m^{\nu} - \ell^{\nu}m^{\mu})(\partial_{\mu}\ell'_{\nu}) = 0 \quad , \quad (\ell^{\mu}m^{\nu} - \ell^{\nu}m^{\mu})(\partial_{\mu}m'_{\nu}) = 0 \\
\\
&(n^{\mu}\overline{m}^{\nu} - n^{\nu}\overline{m}^{\mu})(\partial_{\mu}n'_{\nu}) = 0 \quad , \quad (n^{\mu}\overline{m}^{\nu} - n^{\nu}\overline{m}^{\mu})(\partial_{\mu}\overline{m'}_{\nu}) = 0
\end{aligned}
\tag{4.3}
$$

which are equivalent to

$$
\begin{aligned}
&d\ell' = Z_1 \wedge \ell' + i\Phi_1 m' \wedge \overline{m'} \\
&dm' = Z_3 \wedge m' + \Phi_3 \ell' \wedge n' \\
\\
&dn' = Z_2 \wedge n' + i\Phi_2 m' \wedge \overline{m'} \\
&d\overline{m'} = \overline{Z_3} \wedge \overline{m'} + \overline{\Phi_3}\ell' \wedge n'
\end{aligned}
\tag{4.4}
$$

where Z_1, Z_2 are real 1-forms, Z_3 a complex 1-form, Φ_1, Φ_2 two real scalars and Φ_3 a complex scalar.

The reader familiar with general relativity must be careful. The present tetrad of vectors (a basis of the tangent space of the manifold) and the corresponding tetrad of 1-forms (a basis of the cotangent space of the manifold) are not orthonormal, because simply we have not yet assumed any "metric" in the tangent and cotangent vector spaces. This first notion of linearly independent "moving" frame was introduced by Elie Cartan and it is well understood in his formalism. Therefore, I will remove the primes on the 1-forms. General relativity (Einstein's riemannian geometry) is based on the introduction of the metric structure η_{ab} in the cotangent space. Instead, I introduce the above structure, which I call lorentzian CR-structure (LCR-structure). The ambition of PCFT is to derive the dynamics of the leptonic sector from these conditions. Notice that it coincides with the geodetic and shear-free conditions on the $\ell^{\mu}\partial_{\mu}$ and $n^{\mu}\partial_{\mu}$ null congruences of general relativity. We will also see that the differences of manifolds (LCR-manifolds and lorentzian riemannian manifolds) endowed with these two fundamental structures are essential.

The LCR-structure is **not** invariant under the internal local $SO(1,3)$ symmetry of Einstein's riemannian geometry. Instead it admits the transformation

$$
\begin{aligned}
&\ell'_{\mu} = \Lambda\ell_{\mu} \quad , \quad \ell'^{\mu} = \tfrac{1}{N}\ell^{\mu} \\
&n'_{\mu} = Nn_{\mu} \quad , \quad n'^{\mu} = \tfrac{1}{\Lambda}n^{\mu} \\
&m'_{\mu} = Mm_{\mu} \quad , \quad m'^{\mu} = \tfrac{1}{\overline{M}}m^{\mu}
\end{aligned}
\tag{4.5}
$$

which I call tetrad-Weyl transformation, because the non-vanishing Weyl factors are applied to the LCR-tetrad (not all the tetrads), than to the metric tensor as it is the case of the ordinary Weyl transformation. The auxiliary fields transform as follows

$$
\begin{array}{lll}
Z'_{1\mu} = Z_{1\mu} + \partial_\mu \ln \Lambda \ , & Z'_{2\mu} = Z_{2\mu} + \partial_\mu \ln N \ , & Z'_{3\mu} = Z_{3\mu} + \partial_\mu \ln M \\
\Phi'_1 = \frac{\Lambda}{M\overline{M}}\Phi_1 \quad , & \Phi'_2 = \frac{N}{M\overline{M}}\Phi_2 \quad , & \Phi'_3 = \frac{M}{\Lambda N}\Phi_3
\end{array}
\tag{4.6}
$$

Notice that if Φ_1, Φ_2 and Φ_3 do not vanish, we can always make a tetrad-Weyl transformation which fixes them to 1. That is these scalar constants take the values either zero or one. They act as topological invariants, which may stabilize solitonic configurations. Also notice that

$$
F_1 = dZ_1 \quad , \quad F_2 = dZ_2 \quad , \quad F_3 = dZ_3 \tag{4.7}
$$

are LCR invariant.

It is evident that the abelian gauge fields Z_1, Z_2 and Z_3 are defined up to a term proportional to ℓ_μ, n_μ and m_μ respectively. Using the NP-coefficients they take the form

$$
\begin{array}{l}
Z_{1\mu} = (\theta_1 + \mu + \overline{\mu})\ell_\mu + (\varepsilon + \overline{\varepsilon})n_\mu - (\alpha + \overline{\beta} - \overline{\tau})m_\mu - \\
\qquad -(\overline{\alpha} + \beta - \tau)\overline{m}_\mu \\
Z_{2\mu} = -(\gamma + \overline{\gamma})\ell_\mu + (\theta_2 - \rho - \overline{\rho})n_\mu - (\pi - \alpha - \overline{\beta})m_\mu - \\
\qquad -(\overline{\pi} - \overline{\alpha} - \beta)\overline{m}_\mu \\
Z_{3\mu} = (\gamma - \overline{\gamma} + \overline{\mu})\ell_\mu + (\varepsilon - \overline{\varepsilon} - \rho)n_\mu - (\theta_3 + \pi - \overline{\tau})m_\mu - \\
\qquad -(\beta - \overline{\alpha})\overline{m}_\mu \\
\Phi_1 = \frac{\rho - \overline{\rho}}{i} \quad , \quad \Phi_2 = \frac{\mu - \overline{\mu}}{i} \quad , \quad \Phi = -(\tau + \overline{\pi})
\end{array}
\tag{4.8}
$$

with the functions θ_1 , θ_2 , θ_3 á priori arbitrary. These arbitrary terms may be fixed, using the following transformations of the NP-coefficients

$$
\begin{array}{l}
\alpha' = \frac{1}{M}\alpha + \frac{M\ \overline{M} - \Lambda N}{4M\Lambda N}(\overline{\tau} + \pi) + \frac{1}{4M}\overline{\delta}\ln\frac{\Lambda}{N\overline{M}^2} \\
\beta' = \frac{1}{\overline{M}}\beta + \frac{M\ \overline{M} - \Lambda N}{4\overline{M}\Lambda N}(\tau + \overline{\pi}) + \frac{1}{4\overline{M}}\delta\ln\frac{\Lambda M^2}{N} \\
\gamma' = \frac{1}{\Lambda}\gamma + \frac{M\ \overline{M} - \Lambda N}{4M\ \overline{M}\Lambda}(\overline{\mu} - \mu) + \frac{1}{4\Lambda}\Delta\ln\frac{M}{N^2\overline{M}} \\
\varepsilon' = \frac{1}{N}\varepsilon + \frac{M\ \overline{M} - \Lambda N}{4M\ \overline{M}N}(\overline{\rho} - \rho) + \frac{1}{4N}D\ln\frac{M\Lambda^2}{\overline{M}} \\
\mu' = \frac{1}{2\Lambda}(\mu + \overline{\mu}) + \frac{N}{2M\ \overline{M}}(\mu - \overline{\mu}) + \frac{1}{2\Lambda}\Delta\ln(M\ \overline{M}) \\
\rho' = \frac{1}{2N}(\rho + \overline{\rho}) + \frac{\Lambda}{2M\ \overline{M}}(\rho - \overline{\rho}) - \frac{1}{2N}D\ln(M\ \overline{M}) \\
\pi' = \frac{\overline{M}}{2\Lambda N}(\pi + \overline{\tau}) + \frac{1}{2M}(\pi - \overline{\tau}) + \frac{1}{2M}\overline{\delta}\ln(\Lambda N) \\
\tau' = \frac{M}{2\Lambda N}(\tau + \overline{\pi}) + \frac{1}{2\overline{M}}(\tau - \overline{\pi}) - \frac{1}{2\overline{M}}\delta\ln(\Lambda N) \\
\kappa' = \frac{\Lambda}{N\overline{M}}\kappa \quad , \quad \sigma' = \frac{M}{N\overline{M}}\sigma \\
\nu' = \frac{N}{\Lambda M}\nu \quad , \quad \lambda' = \frac{\overline{M}}{\Lambda M}\lambda
\end{array}
\tag{4.9}
$$

under a tetrad-Weyl transformation. In the generic case of non-vanishing relative invariants, the transformations are satisfied if the additional terms are

$$
\theta_1 = n^\mu \partial_\mu \ln\frac{\rho - \overline{\rho}}{i} \quad , \quad \theta_2 = \ell^\mu \partial_\mu \ln\frac{\mu - \overline{\mu}}{-i} \quad , \quad \theta_3 = \overline{m}^\mu \partial_\mu \ln(-\tau - \overline{\pi}) \tag{4.10}
$$

In brief, the LCR-structure is the fundamental geometric structure and it is the origin of the Newman "magic recesses" ([27]). The purpose of the present Research eBook is to show how all the interactions and quantum theory itself emerge.

4.1 LCR-structure coordinates

The LCR-structure conditions are simply the necessary hypothesis to apply the holomorphic Frobenius theorem, without the existence of a precise metric structure. The holomorphic version of the theorem is imposed by the complex nature of one vector (m) of the pair. The application of this theorem implies the existence of a generally complex coordinate system $(z^\alpha, z^{\widetilde{\beta}})\ :\ \alpha,\beta = 0,1$, such that

$$\begin{aligned}
&dz^\alpha = f_0^\alpha\ \ell_\mu dx^\mu + f_1^\alpha\ m_\mu dx^\mu \quad , \quad dz^{\widetilde{\alpha}} = f_{\widetilde{0}}^{\widetilde{\alpha}}\ n_\mu dx^\mu + f_{\widetilde{1}}^{\widetilde{\alpha}}\ \widetilde{m}_\mu dx^\mu \\
&\ell = \ell_\alpha dz^\alpha\ ,\ m = m_\alpha dz^\alpha \quad , \quad n = n_{\widetilde{\alpha}} dz^{\widetilde{\alpha}}\ ,\ \widetilde{m} = \widetilde{m}_{\widetilde{\alpha}} dz^{\widetilde{\alpha}}
\end{aligned} \tag{4.11}$$

where (ℓ, m) and $(n, \widetilde{m})$ are the pairs of the cotangent tetrad after the necessary complexification of the coordinates x^μ to $r^\mu = x^\mu + iy^\mu$. By construction, the coordinate functions $(z^\alpha(r), z^{\widetilde{\beta}}(r))$ determine a holomorphic transformation in a patch of $\mathbb{C}^4$ outside the real surface $\mathrm{Im}(r) = 0$, which is the LCR-manifold M, viewed as a real submanifold of the ambient complex manifold[1]. When we return in M, the generally complex functions $(z^\alpha(x), z^{\widetilde{\beta}}(x))|_M$ may become generalized functions with singular support at the points where they are not real analytic. These are the points x, where $(z^\alpha(x), z^{\widetilde{\beta}}(x))$ is not analytic in both sides of the real surface. Or vice-versa, at the singular points on the real surface, the structure coordinates $(z^\alpha(x), z^{\widetilde{\beta}}(x))$ can be analytically extended towards the one side of the real surface, but not in the other.

After the Lewy remark, that a typical partial differential equation (PDE) of CR-structures does not have non-constant solution, the mathematicians consider the possibility of non-existence of CR-structure coordinates. We will not consider such non-realizable LCR-structures.

The fact that the ℓ and n are real and m is complex implies the following conditions of $(z^\alpha(x), z^{\widetilde{\beta}}(x))|_M$,

$$\begin{aligned}
&dz^0 \wedge dz^1 \wedge d\overline{z^0} \wedge d\overline{z^1} = 0 \\
&dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \wedge d\overline{z^0} \wedge d\overline{z^1} = 0 \\
&dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \wedge d\overline{z^{\widetilde{0}}} \wedge d\overline{z^{\widetilde{1}}} = 0 \\
&\\
&dz^0 \wedge dz^1 \wedge dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \neq 0
\end{aligned} \tag{4.12}$$

that is, there are two real functions ρ_{11} , ρ_{22} and a complex one ρ_{12}, such that

$$\begin{aligned}
&\rho_{11}(\overline{z^\alpha}, z^\alpha) = 0 \quad , \quad \rho_{12}\left(\overline{z^\alpha}, z^{\widetilde{\alpha}}\right) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\alpha}}) = 0 \\
&\tfrac{\partial \rho_{ij}}{\partial z^b} \neq 0 \neq \tfrac{\partial \rho_{ij}}{\partial \overline{z^b}}
\end{aligned} \tag{4.13}$$

These functions are defined up to non-vanishing factors. According to the conventional terminology, the manifold is locally (in every patch of a covering atlas) a totally real submanifold of $\mathbb{C}^4$. Notice that the defining functions do not depend on all the structure coordinates. The precise dependence of the defining functions on the structure coordinates characterizes the LCR-structure from the general definition of a totally real submanifold of $\mathbb{C}^4$. The four functions $z^b \equiv (z^\alpha,\ z^{\widetilde{\alpha}})$, $\alpha = 0,\ 1$ are the structure coordinates of the LCR-structure in the corresponding coordinate chart. The holomorphic transformations in the intersection of the charts (of the LCR-atlas), which preserve the LCR-structure, are

$$z'^\alpha = f^\alpha(z^\beta) \quad , \quad z'^{\widetilde{\alpha}} = f^{\widetilde{\alpha}}(z^{\widetilde{\beta}}) \tag{4.14}$$

which will be called LCR-transformations. I point out that the general holomorphic transformations $z'^b = f^b(z^c)$ do not preserve the LCR-structure! In a neighborhood of a point p, a LCR-transformation can simplify[1] a smooth structure to the form

$$\begin{array}{l}
\operatorname{Im} z^0 = \phi_{11}(\overline{z^1}, z^1, \operatorname{Re} z^0)\ ,\ \operatorname{Im} z^{\widetilde{0}} = \phi_{22}(\overline{z^{\widetilde{1}}}, z^{\widetilde{1}}, \operatorname{Re} z^{\widetilde{0}})\ ,\ z^{\widetilde{1}} - \overline{z^1} = \phi_{12}(\overline{z^a}, z^{\widetilde{0}}) \\[2ex]
\phi_{11}(p) = \phi_{22}(p) = \phi_{12}(p) = 0 \quad , \quad d\phi_{11}(p) = d\phi_{22}(p) = d\phi_{12}(p) = 0
\end{array} \tag{4.15}$$

and the corresponding coordinates are called regular LCR-coordinates in the neighborhood of the point p. The LCR-transformations cannot completely remove (annihilate) the real analytic functions ϕ_{ij}. But, in the neighborhood of a real analytic point p, an **ordinary** holomorphic transformation $z'^b = f^b(z^c)$ can remove these functions. That is, in the neighborhood of a real analytic point p, a holomorphic transformation makes a real analytic LCR-structure equivalent to the degenerate totally real CR-structure, which cannot be generally done with a LCR-transformation. Hence at points p with a real analytic neighborhood, there are general complex coordinates r^b, $b = 0, 1, 2, 3$, such that $r^b - \overline{r^b} = 0$. From this mathematical subtlety we conclude that the structure coordinates are singular generalized functions $z^a(x^b)$ on M, which satisfy the compatible conditions

$$\begin{array}{lcl}
\ell^\mu \partial_\mu z^\beta = 0 & , & m^\mu \partial_\mu z^\beta = 0 \\
n^\mu \partial_\mu z^{\widetilde{\beta}} = 0 & , & \overline{m}^\mu \partial_\mu z^{\widetilde{\beta}} = 0
\end{array} \tag{4.16}$$

The inverse procedure to find a tetrad $(\ell\ ,\ n\ ,\ m\ ,\ \overline{m})$ from the defining LCR-structure conditions (4.13) is straightforward. It is convenient to use the notation $\partial' f = \frac{\partial f}{\partial z^\alpha} dz^\alpha$ and $\partial'' f = \frac{\partial f}{\partial z^{\widetilde{\alpha}}} dz^{\widetilde{\alpha}}$. Because of $d\rho_{ij} = 0$ and the special dependence of each function on the structure coordinates $\left(z^\alpha, z^{\widetilde{\alpha}}\right)$, we find

$$\begin{array}{l}
\ell = 2i\partial\rho_{11} = -2i\overline{\partial}\rho_{11} = 2i\partial'\rho_{11} = i(\partial' - \overline{\partial'})\rho_{11} \\
n = 2i\partial\rho_{22} = -2i\overline{\partial}\rho_{22} = 2i\partial''\rho_{22} = i(\partial'' - \overline{\partial''})\rho_{22} \\
m_1 = 2i\partial\frac{\rho_{12}+\overline{\rho_{12}}}{2} = -2i\overline{\partial}\frac{\rho_{12}+\overline{\rho_{12}}}{2} = i(\partial - \overline{\partial})\frac{\rho_{12}+\overline{\rho_{12}}}{2} = i(\partial' + \partial'' - \overline{\partial'} - \overline{\partial''})\frac{\rho_{12}+\overline{\rho_{12}}}{2} \\
m_2 = 2i\partial\frac{\overline{\rho_{12}}-\rho_{12}}{2i} = -2i\overline{\partial}\frac{\overline{\rho_{12}}-\rho_{12}}{2i} = i(\partial - \overline{\partial})\frac{\overline{\rho_{12}}-\rho_{12}}{2i} = i(\partial' + \partial'' - \overline{\partial'} - \overline{\partial''})\frac{\overline{\rho_{12}}-\rho_{12}}{2i}
\end{array} \tag{4.17}$$

where we consider all these differential 1-forms restricted on the defined submanifold, therefore they are real. The relations become simpler, if we use the complex 1-form

$$\begin{array}{l} m = m_1 + im_2 = 2i\partial\overline{\rho_{12}} = -2i\overline{\partial}\overline{\rho_{12}} = i(\partial - \overline{\partial})\overline{\rho_{12}} \\ \overline{m} = m_1 - im_2 = 2i\partial\rho_{12} = -2i\overline{\partial}\rho_{12} = i(\partial - \overline{\partial})\rho_{12} \end{array} \tag{4.18}$$

This tetrad of M is apparently defined up to a tetrad-Weyl transformation implied by the ambiguity of ρ_{ij} with non-vanishing factors.

4.2 Examples of LCR-structures

The light-cone coordinates is the simplest example of LCR-structure in $\mathbb{R}^4$. The LCR-tetrad and the corresponding structure coordinates are

$$\begin{array}{l} \ell = dx^0 - dx^3 \quad , \quad m = dx^1 + idx^2 \quad , \quad n = dx^0 + dx^3 \\ z^0 = x^0 - x^3 \quad , \quad z^1 = x^1 + ix^2 \quad , \quad z^{\widetilde{0}} = x^0 + x^3 \quad , \quad z^{\widetilde{1}} = x^1 - ix^2 \\ \\ \ell \wedge m \wedge n \wedge \overline{m} = 4idx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \neq 0 \quad \forall x^\mu \in \mathbb{R}^4 \end{array} \tag{4.19}$$

I point out that the differential forms are coordinate independent structures. At the intersection of two charts, they have two equivalent expressions derived the one from the other according to the coordinate transformations. In order to compare two forms, we have to write them in the same coordinate chart. We will now compare the above degenerate LCR-structure with the following spherical light-cone LCR-structure

$$\begin{array}{l} \ell' = dx^0 - dr = dx^0 - \frac{1}{r}(x^1dx^1 + x^2dx^2 + x^3dx^3) \\ m' = d(\tan\frac{\theta}{2}e^{i\varphi}) = d\frac{x^1+ix^2}{x^3+r} = \frac{1}{r(x^3+r)^2}[(r(x^3+r) - (x^1+ix^2)x^1)dx^1 + \\ \quad +(ir(x^3+r) - (x^1+ix^2)x^2)dx^2 - (x^1+ix^2)(x^3+r)dx^3] \\ n' = dx^0 + dr = dx^0 + \frac{1}{r}(x^1dx^1 + x^2dx^2 + x^3dx^3) \\ z'^0 = x^0 - r \quad , \quad z'^1 = \frac{x^1+ix^2}{x^3+r} \quad , \quad z'^{\widetilde{0}} = x^0 + r \quad , \quad z'^{\widetilde{1}} = \frac{x^1-ix^2}{x^3+r} \\ \\ \ell' \wedge m' \wedge n' \wedge \overline{m}' = \frac{4i}{(x^3+r)^2}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \neq 0, \quad \forall x^\mu \in \mathbb{R}^4 - \{\mathbb{R}_-\} \end{array} \tag{4.20}$$

The tetrad is singular at $x^3 + r = 0$, that is the negative z-axis $\{x^1 = 0 = x^2,\ x^3 \leq 0\}$. These singularities are not removable, that is, they are not absorbed by a **singular** tetrad-Weyl transformation. Hence we conclude that the above light-cone LCR-structures are not equivalent. One can easily see that by simply noticing that the relation

$$dz'^0 = \frac{x^3+r}{2r}dz^0 - \frac{x^1-ix^2}{2r}dz^1 + \frac{r-x^3}{2r}dz^{\widetilde{0}} - \frac{x^1+ix^2}{2r}dz^{\widetilde{1}} \tag{4.21}$$

is not LCR compatible.

Let us now consider the "Schwartzschild" LCR-tetrad

$$\begin{array}{l} \ell_\mu dx^\mu = (r-2M)dt - rdr \quad , \quad n_\mu dx^\mu = (r-2M)dt + rdr \\ m_\mu dx^\mu = d\theta + i\sin\theta d\varphi \\ \\ \ell\wedge m\wedge n\wedge \overline{m} = 4ir(r-2M)\sin\theta dt\wedge dr\wedge d\theta\wedge d\varphi \neq 0 \\ r\neq 0 \quad , \quad r\neq 2M \quad , \quad \sin\theta\neq 0 \end{array} \tag{4.22}$$

where the relations of the coordinates (t, r, θ, φ) with the cartesian coordinates is not straightforward. Its structure coordinates are

$$\begin{array}{ll} z^0 = t - r - 2M\ln\frac{|r-2M|}{r_0} \quad , & z^1 = e^{i\varphi}\tan\frac{\theta}{2} \\ z^{\widetilde{0}} = t + r + 2M\ln\frac{|r-2M|}{r_0} \quad , & z^{\widetilde{1}} = e^{-i\varphi}\tan\frac{\theta}{2} \end{array} \tag{4.23}$$

where r_0 is a normalization. All the previous LCR-structures are degenerate ($\equiv$ all its relative invariants Φ_1, Φ_2, Φ vanish), but apparently they are not LCR-equivalent.

The symmetric "Kerr-Newman" LCR-tetrad is

$$\begin{array}{l} \ell_\mu dx^\mu = \Delta dt - \eta\overline{\eta}dr - a\Delta\sin^2\theta d\varphi \\ n_\mu dx^\mu = \Delta dt + \eta\overline{\eta}dr - a\Delta\sin^2\theta d\varphi \\ m_\mu dx^\mu = ia\sin\theta dt - \rho^2 d\theta - i(r^2+a^2)\sin\theta d\varphi \\ \ell\wedge m\wedge n\wedge \overline{m} = 4i\eta^3\overline{\eta}^3\Delta\sin\theta dt\wedge dr\wedge d\theta\wedge d\varphi\neq 0 \\ \Delta := r^2 - 2Mr + a^2 + q^2 \neq 0 \quad , \quad \eta := r + ia\cos\theta\neq 0 \quad , \quad \sin\theta\neq 0 \end{array} \tag{4.24}$$

Its structure coordinates are

$$\begin{array}{l} z^0 = t - f_0(r) + ia\cos\theta - ia \quad , \quad z^1 = e^{i\varphi}e^{-iaf_1(r)}\tan\frac{\theta}{2} \\ z^{\widetilde{0}} = t + f_0(r) - ia\cos\theta + ia \quad , \quad z^{\widetilde{1}} = e^{-i\varphi}e^{-iaf_1(r)}\tan\frac{\theta}{2} \\ \\ f_0(r) = \int\frac{r^2+a^2}{\Delta}dr \quad , \quad f_1(r) = \int\frac{1}{\Delta}dr \end{array} \tag{4.25}$$

The relative invariants of this LCR-structure do not vanish, and they are proportional to a.

The "Taub-NUT" LCR-tetrad is

$$\begin{array}{l} \ell_\mu dx^\mu = fdt - (r^2+l^2)dr + 4lf\sin^2\frac{\theta}{2}\, d\varphi \\ n_\mu dx^\mu = fdt + (r^2+l^2)dr + 4lf\sin^2\frac{\theta}{2}\, d\varphi \\ m_\mu dx^\mu = d\theta + i\sin\theta d\varphi \\ \ell\wedge m\wedge n\wedge \overline{m} = 4if(r^2+l^2)\sin\theta dt\wedge dr\wedge d\theta\wedge d\varphi\neq 0 \\ f = r^2 - 2Mr - l^2 \neq 0 \quad , \quad \sin\theta\neq 0,\ \pi \end{array} \tag{4.26}$$

which is singular at $f = 0$. This LCR-structure has the relative invariants $\Phi_1 \neq 0 \neq \Phi_2$ and $\Phi = 0$. The structure coordinates are

$$\begin{array}{l} z^0 = t - r' - 4il\ln(\cos\frac{\theta}{2}) \quad , \quad z^1 = e^{i\varphi}\tan\frac{\theta}{2} \\ \\ z^{\widetilde{0}} = t + r' - 4il\ln(\cos\frac{\theta}{2}) \quad , \quad z^{\widetilde{1}} = e^{-i\varphi}\tan\frac{\theta}{2} \\ \\ r' = \int\frac{dr}{f} \end{array} \tag{4.27}$$

Considering the following LCR-structure preserving transformations

$$\begin{array}{ll}
i\frac{z^0}{4l}=\ln(\cos\frac{\theta}{2}e^{i\frac{t-r'}{4l}})=\ln w^0 \quad , & z^1=e^{i\varphi}\tan\frac{\theta}{2}=\frac{w^1}{w^0} \\
i\frac{z^{\widetilde{0}}}{2l}=\ln(\cos\frac{\theta}{2}e^{i\frac{t+r'}{4l}})=\ln w^{\widetilde{0}} \quad , & z^{\widetilde{1}}=e^{-i\varphi}\tan\frac{\theta}{2}=\frac{w^{\widetilde{1}}}{w^{\widetilde{0}}}
\end{array} \tag{4.28}$$

the new variables w^a satisfy the embedding functions

$$\begin{array}{l}
\rho_{11}=w^0\overline{w^0}+w^1\overline{w^1}-1=0 \\
\rho_{12}=\overline{w^0}w^{\widetilde{1}}-w^{\widetilde{0}}\overline{w^1}=0 \\
\rho_{22}=w^{\widetilde{0}}\overline{w^{\widetilde{0}}}+w^{\widetilde{1}}\overline{w^{\widetilde{1}}}-1=0
\end{array} \tag{4.29}$$

which are equivalent to the natural LCR-structure of $U(2)$, which we derive below.

We now consider the group manifold $U(1)\times SU(2)[=S^1\times S^3]$ with generators

$$\begin{array}{l}
\sigma^0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma^1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma^2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\\
[\sigma^0\ ,\ \sigma^j]=0 \quad , \quad [\sigma^i\ ,\ \sigma^j]=2i\epsilon_{ijk}\sigma^k
\end{array} \tag{4.30}$$

Its ordinary parametrization

$$\begin{array}{l}
U=e^{i\tau}\begin{pmatrix} \cos\rho+i\sin\rho\cos\theta & -i\sin\rho\sin\theta\ e^{-i\varphi} \\ -i\sin\rho\sin\theta\ e^{i\varphi} & \cos\rho-i\sin\rho\cos\theta \end{pmatrix} \\
\tau\in(0,2\pi) \quad , \quad \rho\in[0,2\pi) \quad , \quad \theta\in[0,\pi] \quad , \quad \varphi\in[0,2\pi)
\end{array} \tag{4.31}$$

The 1-forms of its left invariant generators $e_L^a=e_{L\mu}^a dx^\mu$ (a basis of the cotangent space) defined by the relation $U^\dagger dU=ie_L^a\sigma_a=i(e_L^0\sigma^0-e_L^i\sigma^i)$, implies

$$\begin{array}{l}
e_L^0=d\tau \\
e_L^1=\sin\theta\cos\varphi d\rho+\sin\rho(\sin\rho\sin\varphi+\cos\rho\cos\theta\cos\varphi)d\theta- \\
\qquad -\sin\rho\sin\theta(\sin\rho\cos\theta\cos\varphi+\cos\rho\sin\varphi)d\varphi \\
e_L^2=\sin\theta\sin\varphi d\rho-\sin\rho(\cos\rho\cos\theta\sin\varphi+\sin\rho\cos\varphi)d\theta- \\
\qquad -\sin\rho\sin\theta(\sin\rho\cos\theta\sin\varphi-\cos\rho\cos\varphi)d\varphi \\
e_L^3=-\cos\theta d\rho+\sin\rho\cos\rho\sin\theta d\theta-\sin^2\rho\sin^2\theta d\varphi
\end{array} \tag{4.32}$$

The corresponding LCR-tetrad and the $U(2)$ group LCR-structure equations take the following appropriate form

$$\begin{array}{l}
\omega=U^{-1}dU=:i\begin{pmatrix} \ell & \overline{m} \\ m & n \end{pmatrix} \quad , \quad d\omega+\omega\wedge\omega=0 \\
\\
d\ell=im\wedge\overline{m} \quad , \quad dn=-im\wedge\overline{m} \quad , \quad dm=i(\ell-n)\wedge m
\end{array} \tag{4.33}$$

The LCR-structure coordinates are

$$\begin{array}{lll} w^0 = (\cos\rho + i\sin\rho\cos\theta)e^{i\tau} & , & w^1 = \sin\rho\sin\theta\ e^{i\varphi}e^{i\tau} \\ w^{\widetilde{0}} = (\cos\rho - i\sin\rho\cos\theta)e^{i\tau} & , & w^{\widetilde{1}} = \sin\rho\sin\theta\ e^{-i\varphi}e^{i\tau} \end{array} \tag{4.34}$$

which satisfy the embedding relations (4.29) of the Taub-NUT LCR-structure into the ambient complex manifold.

5 THE 4-D PSEUDO-CONFORMAL LAGRANGIANS

We saw that the 4-dimensional spacetime metrics cannot generally take an off-diagonal form analogous to the 2-dimensional metrics. Only metrics, which admit two geodetic and shear-free null congruences $\ell^\mu\partial_\mu$, $n^\mu\partial_\mu$ can take this form

$$ds^2 = 2g_{\alpha\widetilde{\beta}}dz^\alpha dz^{\widetilde{\beta}} \quad , \quad \alpha, \widetilde{\beta} = 0, 1 \tag{5.1}$$

where the LCR-structure coordinates $z^b = (z^\alpha(x), z^{\widetilde{\beta}}(x))$ are generally complex functions. In this case we can write down the following metric independent Yang-Mills-like integral

$$\begin{array}{l} I_G = \int d^4z\ \sqrt{-g}g^{\alpha\widetilde{\alpha}}g^{\beta\widetilde{\beta}}F_{j\alpha\beta}F_{j\widetilde{\alpha}\widetilde{\beta}} = \int d^4z\ F_{j01}F_{j\widetilde{0}\widetilde{1}} \\ \\ F_{j_{ab}} = \partial_a A_{jb} - \partial_a A_{jb} - \gamma\, f_{jik}A_{ia}A_{kb} \end{array} \tag{5.2}$$

which depends on the LCR-structure coordinates $(z^\alpha(x), z^{\widetilde{\beta}}(x))$, and it does not depend on the metric. This property is completely analogous to that of Polyakov action. This integral is apparently complex, because the structure coordinates are complex. Therefore the real spacetime action must be either its real or imaginary part. The restriction on the metrics which admit two geodetic and shear-free congruences, should not physically bother us, because the black-holes have this property. On the contrary, it is rather encouraging, because it provides an argument why all the observed spacetimes are Schwartzschild type.

The integral (5.2) is complex and not generally covariant. It is written in the LCR-structure (chiral) coordinates (where the metric independence appears) in order to clarify how the metric independence of the Polyakov action triggered the search, discovery and study of the dynamical content of the 4-dimentional PCFT.

The fact that the structure coordinates are generally complex implies that the original metric independent form (5.2) is complex, while the final action must be real. In order to make things clear, I will start from the LCR compatible gauge connection and its curvature

$$\begin{array}{lll} (D_\alpha)_{ij} = \partial_\alpha\delta_{ij} - \gamma f_{ikj}A_{k\alpha} & , & (D_{\widetilde{\beta}})_{ij} = \partial_{\widetilde{\beta}}\delta_{ij} - \gamma f_{ikj}A_{k\widetilde{\beta}} \\ F_{i\alpha\beta} = \partial_\alpha A_{i\beta} - \partial_\beta A_{i\alpha} - \gamma f_{ikj}A_{j\alpha}A_{k\beta} & , & F_{i\widetilde{\alpha}\widetilde{\beta}} = \partial_{\widetilde{\alpha}}A_{i\widetilde{\beta}} - \partial_{\widetilde{\beta}}A_{i\widetilde{\alpha}} - \gamma f_{ikj}A_{j\widetilde{\alpha}}A_{k\widetilde{\beta}} \end{array} \tag{5.3}$$

in structure coordinates. The gauge invariant and metric independent 4-form is

$$F \wedge \widetilde{F} = (\tfrac{1}{2} F_{i\alpha\beta} dz^{\alpha} \wedge dz^{\beta}) \wedge (\tfrac{1}{2} F_{i\widetilde{\alpha}\widetilde{\beta}} dz^{\widetilde{\alpha}} \wedge dz^{\widetilde{\beta}}) = F_{i01} F_{i\widetilde{0}\widetilde{1}} dz^{0} \wedge dz^{1} \wedge dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \tag{5.4}$$

Using the identity

$$\begin{aligned} &\delta^{\mu}_{\nu} = \ell^{\mu} n_{\nu} + n^{\mu} \ell_{\nu} - m^{\mu} \widetilde{m}_{\nu} - \widetilde{m}^{\mu} m_{\nu} \\ &\delta^{\alpha}_{\beta} = n^{\alpha} \ell_{\beta} - \widetilde{m}^{\alpha} m_{\beta} \quad , \quad \delta^{\widetilde{\alpha}}_{\widetilde{\beta}} = \ell^{\widetilde{\alpha}} n_{\widetilde{\beta}} - m^{\widetilde{\alpha}} \widetilde{m}_{\widetilde{\beta}} \end{aligned} \tag{5.5}$$

in structure coordinates, the complexified 4-form becomes

$$F \wedge \widetilde{F} = \ell \wedge m \wedge n \wedge \widetilde{m} (\ell^{\mu} m^{\nu} F_{i\mu\nu})(n^{\rho} \widetilde{m}^{\sigma} F_{i\rho\sigma}) \tag{5.6}$$

When we return back to the real spacetime, it becomes the complex 4-form

$$\begin{aligned} &(F \wedge \widetilde{F})|_{S} = \ell \wedge m \wedge n \wedge \overline{m} (\ell^{\mu} m^{\nu} F_{i\mu\nu})(n^{\rho} \overline{m}^{\sigma} F_{i\rho\sigma}) \\ &\ell \wedge m \wedge n \wedge \overline{m} = d^4x \sqrt{-g}\, i \\ &g = \det(g_{\mu\nu}) = \det(\eta_{ab})[\det(e^{a}_{\mu})]^2 = [\det(e^{a}_{\mu})]^2 \\ &\eta_{ab} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{aligned} \tag{5.7}$$

Hence we may assume as gauge field action either its real or its imaginary part

$$\begin{aligned} &I_R = \int d^4x \sqrt{-g}\, i \{ (\ell^{\mu} m^{\nu} F_{i\mu\nu})(n^{\rho} \overline{m}^{\sigma} F_{i\rho\sigma}) - (\ell^{\mu} \overline{m}^{\nu} F_{i\mu\nu})(n^{\rho} m^{\sigma} F_{i\rho\sigma}) \} \\ &I_I = \int d^4x \sqrt{-g} \{ (\ell^{\mu} m^{\nu} F_{i\mu\nu})(n^{\rho} \overline{m}^{\sigma} F_{i\rho\sigma}) + (\ell^{\mu} \overline{m}^{\nu} F_{i\mu\nu})(n^{\rho} m^{\sigma} F_{i\rho\sigma}) \} \\ &F_{j\mu\nu} = \partial_{\mu} A_{j\nu} - \partial_{\nu} A_{j\mu} - \gamma\, f_{jik} A_{i\mu} A_{k\nu} \end{aligned} \tag{5.8}$$

Both actions are apparently invariant under the tetrad-Weyl transformation. Notice that only the null self-dual 2-forms appear in the actions. The non-null self-dual component does not appear in the action, because simply it is not multiplicatively transformed relative to the tetrad-Weyl transformation.

In fact these two actions are strongly related. The appearing gauge field tensors $F_{i\mu\nu}$ are each other duals, because $\ell^{[\mu} m^{\nu]}$ and $n^{[\rho} \overline{m}^{\sigma]}$ are self-duals (relative to their corresponding metric). One of these two actions will be the starting point for the emergence of chromodynamics in the context of PCFT.

We saw that the existence of a globally defined LCR-structure is the new (fundamental) mathematical notion, which corresponds to the metric structure of general relativity. In two dimensions all the smooth manifolds are LCR-manifolds, therefore in the Polyakov functional integral we simply integrate over all 2-dimensional manifolds. But in four dimensions we have to consider only the LCR-manifolds. The simple way to impose this restriction is to use the Lagrange multiplier technique to add the following action term with the integrability conditions (4.3) on the tetrad

$$\begin{aligned} I_C = \int d^4x\, \sqrt{-g}\{ & \phi_0(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu)+ \\ & +\phi_1(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) + \phi_{\widetilde{0}}(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu n_\nu)+ \\ & +\phi_{\widetilde{1}}(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu \overline{m}_\nu) + c.conj.\} \end{aligned} \tag{5.9}$$

These Lagrange multipliers make the complete action $I = I_{R(I)} + I_C$ self-consistent and the usual quantization techniques may be applied. The action is formally renormalizable, because it is dimensionless and metric independent. The path-integral quantization of PCFT is also formulated as functional summation of open and closed 4-dimensional LCR-manifolds in complete analogy to the summation of 2-dimensional surfaces in string theory (cobordism procedure). These transition amplitudes of a quantum theory of LCR-manifolds provide (in principle) the self-consistent algorithms for the computation of the physical quantities.

The field equations are formally derived as usual. The gauge field equations are completely different to the ordinary gauge field equations, giving the possibility to find distributional solitonic configurations, which could be identified with the quarks. The impressing feature of the action is the decoupling of the LCR-structure conditions (4.3) from the gauge field equations, which provides the basis of the observed correspondence of the leptonic solitons with the quarks!

6 FIELD EQUATIONS AND INTEGRABILITY CONDITIONS

Variation of the action I_I with respect to the gauge field $A_{j\mu}$ gives the field equations

$$\begin{gathered} D_\mu\{\sqrt{-g}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho \overline{m}^\sigma F_{j\rho\sigma}) + (n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\ell^\rho m^\sigma F_{j\rho\sigma})+ \\ +(\ell^\mu \overline{m}^\nu - \ell^\nu \overline{m}^\mu)(n^\rho m^\sigma F_{j\rho\sigma}) + (n^\mu m^\nu - n^\nu m^\mu)(\ell^\rho \overline{m}^\sigma F_{j\rho\sigma})]\} = 0 \end{gathered} \tag{6.1}$$

where $D_\mu = \delta_{\ell j}\partial_\mu + \gamma f_{\ell jk} A_{k\mu}$ is the gauge symmetry covariant derivative and γ the coupling constant. Multiplying with the null tetrad, these equations take the form

$$\begin{aligned} & m^\mu D_\mu(\ell \overline{m} F_j) + \overline{m}^\mu D_\mu(\ell m F_j) + (\ell \overline{m} F_j)[(\nabla_\mu m^\mu) + (nm\partial\ell)]+ \\ & \quad +(\ell m F_j)[(\nabla_\mu \overline{m}^\mu) + (n\overline{m}\partial\ell)] = 0 \\ & \\ & m^\mu D_\mu(n \overline{m} F_j) + \overline{m}^\mu D_\mu(n m F_j) + (n \overline{m} F_j)[(\nabla_\mu m^\mu) + (\ell m\partial n)]+ \\ & \quad +(n m F_j)[(\nabla_\mu \overline{m}^\mu) + (\ell\overline{m}\partial n)] = 0 \\ & \\ & \ell^\mu D_\mu(n m F_j) + n^\mu D_\mu(\ell m F_j) + (n m F_j)[(\nabla_\mu \ell^\mu) + (\ell \overline{m}\partial m)]+ \\ & \quad +(\ell m F_j)[(\nabla_\mu n^\mu) + (n\overline{m}\partial m)] = 0 \end{aligned} \tag{6.2}$$

Variation of the action with respect to the Lagrange multipliers ϕ_0, ϕ_1, $\phi_{\tilde{0}}$, $\phi_{\tilde{1}}$ imply the complex structure integrability conditions on the tetrad (4.3). Variation of the action with respect to the tetrad gives PDEs on the Lagrange multipliers. In order to preserve the relations between the covariant and contravarient forms of the tetrad we will use the identities

$$\begin{aligned} &\delta e_a^\mu = e_a^\lambda[-n^\mu \delta\ell_\lambda - \ell^\mu \delta n_\lambda + \overline{m}^\mu \delta m_\lambda + m^\mu \delta\overline{m}_\lambda] \\ &\delta\sqrt{-g} = \sqrt{-g}[n^\lambda \delta\ell_\lambda + \ell^\lambda \delta n_\lambda - \overline{m}^\lambda \delta m_\lambda - m^\lambda \delta\overline{m}_\lambda] \end{aligned} \tag{6.3}$$

where we denote ($e_\mu^0 = \ell_\mu$, $e_\mu^1 = m_\mu$) and ($e_\mu^{\tilde{0}} = n_\mu$, $e_\mu^{\tilde{1}} = \overline{m}_\mu$). Variation with respect to ℓ_λ gives the PDEs

$$\begin{aligned} &2\ell^\lambda(nmF_j)(n\overline{m}F_j) + m^\lambda(\ell nF_j)(n\overline{m}F_j) + \overline{m}^\lambda(\ell nF_j)(nmF_j) = \\ &\quad = -\nabla_\mu\left[\phi_0(\ell^\mu m^\lambda - \ell^\lambda m^\mu)\right] - \nabla_\mu\left[\overline{\phi_0}(\ell^\mu\overline{m}^\lambda - \ell^\lambda\overline{m}^\mu)\right] - \\ &\qquad -\ell^\lambda\left[\phi_0(nm\partial\ell) + \overline{\phi_0}(n\overline{m}\partial\ell)\right] - m^\lambda\left[\phi_0(\ell n\partial\ell) + \phi_1(\ell n\partial m)\right] - \\ &\qquad -\overline{m}^\lambda\left[\overline{\phi_0}(\ell n\partial\ell) + \overline{\phi_1}(\ell n\partial\overline{m})\right] \end{aligned} \tag{6.4}$$

which take the tetrad form

$$\begin{aligned} &m^\mu\partial_\mu\phi_0 + \overline{m}^\mu\partial_\mu\overline{\phi_0} + \phi_0[(\nabla_\mu m^\mu) + (\ell m\partial n) - (nm\partial\ell)] + \\ &\quad +\overline{\phi_0}[(\nabla_\mu\overline{m}^\mu) + (\ell\overline{m}\partial n) - (n\overline{m}\partial\ell)] + 2(nmF_j)(n\overline{m}F_j) = 0 \\ &\ell^\mu\partial_\mu\phi_0 + \phi_0[(\nabla_\mu\ell^\mu) + (\ell m\partial\overline{m}) + (\ell n\partial\ell)] + \phi_1(\ell n\partial m) + \\ &\quad +(\ell nF_j)(n\overline{m}F_j) = 0 \end{aligned} \tag{6.5}$$

Variation with respect to n_λ gives the PDEs

$$\begin{aligned} &2n^\lambda(\ell mF_j)(\ell\overline{m}F_j) - m^\lambda(\ell nF_j)(\ell\overline{m}F_j) - \overline{m}^\lambda(\ell nF_j)(\ell mF_j) = \\ &\quad = -\nabla_\mu\left[\phi_{\tilde{0}}(n^\mu\overline{m}^\lambda - n^\lambda\overline{m}^\mu)\right] - \nabla_\mu\left[\overline{\phi_{\tilde{0}}}(n^\mu m^\lambda - n^\lambda m^\mu)\right] - \\ &\qquad -n^\lambda\left[\phi_{\tilde{0}}(\ell\overline{m}\partial n) + \overline{\phi_{\tilde{0}}}(\ell m\partial n)\right] + \overline{m}^\lambda\left[\phi_{\tilde{0}}(\ell n\partial n) + \phi_{\tilde{1}}(\ell n\partial\overline{m})\right] + \\ &\qquad +\overline{m}^\lambda\left[\overline{\phi_{\tilde{0}}}(\ell n\partial\ell) + \overline{\phi_{\tilde{1}}}(\ell n\partial m)\right] \end{aligned} \tag{6.6}$$

which take the tetrad form

$$\begin{aligned} &m^\mu\partial_\mu\phi_0 + \overline{m}^\mu\partial_\mu\overline{\phi_0} + \phi_0[(\nabla_\mu m^\mu) + (\ell m\partial n) - (nm\partial\ell)] + \\ &\quad +\overline{\phi_0}[(\nabla_\mu\overline{m}^\mu) + (\ell\overline{m}\partial n) - (n\overline{m}\partial\ell)] + 2(nmF_j)(n\overline{m}F_j) = 0 \\ &\ell^\mu\partial_\mu\phi_0 + \phi_0[(\nabla_\mu\ell^\mu) + (\ell m\partial\overline{m}) + (\ell n\partial\ell)] + \phi_1(\ell n\partial m) + \\ &\quad +(\ell nF_j)(n\overline{m}F_j) = 0 \end{aligned} \tag{6.7}$$

Variation with respect to m_λ gives the PDEs

$$\begin{aligned} &\ell^\lambda(m\overline{m}F_j)(n\overline{m}F_j) + n^\lambda(m\overline{m}F_j)(\ell\overline{m}F_j) - 2m^\lambda(\ell\overline{m}F_j)(n\overline{m}F_j) = \\ &\quad = -\nabla_\mu\left[\phi_1(\ell^\mu m^\lambda - \ell^\lambda m^\mu)\right] - \nabla_\mu\left[\overline{\phi_{\tilde{1}}}(n^\mu m^\lambda - n^\lambda m^\mu)\right] - \\ &\qquad -\ell^\lambda\left[\phi_0(m\overline{m}\partial\ell) + \phi_1(m\overline{m}\partial m)\right] - n^\lambda\left[\overline{\phi_{\tilde{0}}}(m\overline{m}\partial n) + \overline{\phi_{\tilde{1}}}(m\overline{m}\partial m)\right] + \\ &\qquad +m^\lambda\left[\phi_1(\ell\overline{m}\partial m) + \overline{\phi_{\tilde{1}}}(n\overline{m}\partial m)\right] \end{aligned} \tag{6.8}$$

which take the tetrad form

$$\begin{aligned}
&m^{\mu}\partial_{\mu}\overline{\phi_{\tilde{1}}}+\overline{\phi_{\tilde{1}}}[(\nabla_{\mu}m^{\mu})+(nm\partial\ell)-(m\overline{m}\partial m)]-\overline{\phi_{\tilde{0}}}(m\overline{m}\partial n)-\\
&\qquad-(\ell\overline{m}F_{j})(m\overline{m}F_{j})=0\\[1em]
&m^{\mu}\partial_{\mu}\phi_{1}+\phi_{1}[(\nabla_{\mu}m^{\mu})+(\ell m\partial n)-(m\overline{m}\partial m)]-\phi_{0}(m\overline{m}\partial\ell)-\\
&\qquad-(n\overline{m}F_{j})(m\overline{m}F_{j})=0\\[1em]
&\ell^{\mu}\partial_{\mu}\phi_{1}+n^{\mu}\partial_{\mu}\overline{\phi_{\tilde{1}}}+\phi_{1}[(\nabla_{\mu}\ell^{\mu})+(\ell m\partial\overline{m})-(\ell\overline{m}\partial m)]+\\
&\qquad+\overline{\phi_{\tilde{1}}}[(\nabla_{\mu}n^{\mu})+(nm\partial\overline{m})-(n\overline{m}\partial m)]-2(\ell\overline{m}F_{j})(n\overline{m}F_{j})=0
\end{aligned}\tag{6.9}$$

In order to simplify the relations, I made the bracket notations like $(nm\partial\ell)\equiv(n^{\mu}m^{\nu}-n^{\nu}m^{\mu})\partial_{\mu}\ell_{\nu}$ for the spin coefficients and like $(nmF_{j})\equiv n^{\mu}m^{\nu}F_{j\mu\nu}$ for the gauge field components.

On the other hand the ${e^{a}}_{\mu}$ field equations imply the four conserved currents

$$\begin{aligned}
&\nabla_{\lambda}\{\ell^{\lambda}[2(nmF_{j})(n\overline{m}F_{j})+\phi_{0}(nm\partial\ell)+\overline{\phi_{0}}(n\overline{m}\partial\ell)]+\\
&\qquad+m^{\lambda}[(\ell nF_{j})(n\overline{m}F_{j})+\phi_{0}(\ell n\partial\ell)+\phi_{1}(\ell n\partial m)]+\\
&\qquad+\overline{m}^{\lambda}[(\ell nF_{j})(nmF_{j})+\overline{\phi_{0}}(\ell n\partial\ell)+\overline{\phi_{1}}(\ell n\partial\overline{m})]\}=0\\[1em]
&\nabla_{\lambda}\{n^{\lambda}\left[2(\ell mF_{j})(\ell\overline{m}F_{j})+\phi_{\tilde{0}}(\ell\overline{m}\partial n)+\overline{\phi_{\tilde{0}}}(\ell m\partial n)\right]-\\
&\qquad-\overline{m}^{\lambda}[(\ell nF_{j})(\ell mF_{j})+\phi_{\tilde{0}}(\ell n\partial n)+\phi_{\tilde{1}}(\ell n\partial\overline{m})]-\\
&\qquad-m^{\lambda}\left[(\ell nF_{j})(\ell\overline{m}F_{j})+\overline{\phi_{\tilde{0}}}(\ell n\partial n)+\overline{\phi_{\tilde{1}}}(\ell n\partial m)\right]\}=0\\[1em]
&\nabla_{\lambda}\{\ell^{\lambda}[(m\overline{m}F_{j})(n\overline{m}F_{j})+\phi_{0}(m\overline{m}\partial\ell)+\phi_{1}(m\overline{m}\partial m)]+\\
&\qquad+n^{\lambda}[(m\overline{m}F_{j})(\ell\overline{m}F_{j})+\overline{\phi_{\tilde{0}}}(m\overline{m}\partial n)+\overline{\phi_{\tilde{1}}}(m\overline{m}\partial m)]-\\
&\qquad-m^{\lambda}[2(\ell\overline{m}F_{j})(n\overline{m}F_{j})+\phi_{1}(\ell\overline{m}\partial m)+\overline{\phi_{\tilde{1}}}(n\overline{m}\partial m)]\}=0
\end{aligned}\tag{6.10}$$

These last relations combined with the tetrad integrability conditions imply relations between the surface geometric quantities and the gauge field invariants. For that we will use the following relations of my spin coefficients and the ordinary Newman-Penrose ones

$\alpha=\frac{1}{4}[(\ell n\partial\overline{m})+(\ell\overline{m}\partial n)-(n\overline{m}\partial\ell)-2(m\overline{m}\partial\overline{m})]$
$\beta=\frac{1}{4}[(\ell n\partial m)+(\ell m\partial n)-(nm\partial\ell)-2(m\overline{m}\partial m)]$
$\gamma=\frac{1}{4}[(nm\partial\overline{m})-(n\overline{m}\partial m)-(m\overline{m}\partial n)+2(\ell n\partial n)]$
$\varepsilon=\frac{1}{4}[(\ell m\partial\overline{m})-(\ell\overline{m}\partial m)-(m\overline{m}\partial\ell)+2(\ell n\partial\ell)]$
$\mu=-\frac{1}{2}[(m\overline{m}\partial n)+(nm\partial\overline{m})+(n\overline{m}\partial m)]$
$\pi=\frac{1}{2}[(\ell n\partial\overline{m})-(n\overline{m}\partial\ell)-(\ell\overline{m}\partial n)]$
$\rho=\frac{1}{2}[(\ell\overline{m}\partial m)+(\ell m\partial\overline{m})-(m\overline{m}\partial\ell)]$
$\tau=\frac{1}{2}[(nm\partial\ell)+(\ell m\partial n)+(\ell n\partial m)]$
$\kappa=(\ell m\partial\ell)\quad,\quad\sigma=(\ell m\partial m)$
$\nu=-(n\overline{m}\partial n)\quad,\quad\lambda=-(n\overline{m}\partial\overline{m})$

(6.11)

and the inverse relations

$$\begin{array}{l}(\ell n\partial\ell)=\varepsilon+\overline{\varepsilon}\quad,\quad(\ell m\partial\ell)=\kappa\quad,\quad(nm\partial\ell)=\tau-\overline{\alpha}-\beta\\(\ell n\partial n)=\gamma+\overline{\gamma}\quad,\quad(\ell m\partial n)=\overline{\alpha}+\beta-\overline{\pi}\quad,\quad(nm\partial n)=-\overline{\nu}\\(\ell n\partial m)=\tau+\overline{\pi}\quad,\quad(\ell m\partial m)=\sigma\quad,\quad(\ell\overline{m}\partial m)=\overline{\varepsilon}-\varepsilon+\rho\\(nm\partial m)=-\overline{\lambda}\quad,\quad(n\overline{m}\partial m)=\overline{\gamma}-\gamma-\overline{\mu}\quad,\quad(m\overline{m}\partial m)=\overline{\alpha}-\beta\\(m\overline{m}\partial\ell)=\overline{\rho}-\rho\quad,\quad(m\overline{m}\partial n)=\overline{\mu}-\mu\end{array}\tag{6.12}$$

$$\begin{array}{l}\nabla_\mu\ell^\mu=\varepsilon+\overline{\varepsilon}-\rho-\overline{\rho}\quad,\quad\nabla_\mu n^\mu=\mu+\overline{\mu}-\gamma-\overline{\gamma}\\\nabla_\mu m^\mu=\overline{\pi}+\beta-\tau-\overline{\alpha}\end{array}$$

which are implied by the following formula[6] of the covariant derivatives of the null tetrad

$$\begin{array}{l}\nabla_\mu\ell_\nu=(\gamma+\overline{\gamma})\ell_\mu\ell_\nu-\overline{\tau}\ell_\mu m_\nu-\tau\ell_\mu\overline{m}_\nu+(\varepsilon+\overline{\varepsilon})n_\mu\ell_\nu-\\\qquad-\overline{\kappa}n_\mu m_\nu-\kappa n_\mu\overline{m}_\nu-(\alpha+\overline{\beta})m_\mu\ell_\nu+\overline{\sigma}m_\mu m_\nu+\\\qquad+\rho m_\mu\overline{m}_\nu-(\overline{\alpha}+\beta)\overline{m}_\mu\ell_\nu+\overline{\rho}\overline{m}_\mu m_\nu+\sigma\overline{m}_\mu\overline{m}_\nu\\[1em]\nabla_\mu n_\nu=-(\gamma+\overline{\gamma})\ell_\mu n_\nu+\nu\ell_\mu m_\nu+\overline{\nu}\ell_\mu\overline{m}_\nu-(\varepsilon+\overline{\varepsilon})n_\mu n_\nu+\\\qquad+\pi n_\mu m_\nu+\overline{\pi}n_\mu\overline{m}_\nu+(\alpha+\overline{\beta})m_\mu n_\nu-\lambda m_\mu m_\nu-\\\qquad-\overline{\mu}m_\mu\overline{m}_\nu+(\overline{\alpha}+\beta)\overline{m}_\mu n_\nu-\mu\overline{m}_\mu m_\nu-\overline{\lambda}\overline{m}_\mu\overline{m}_\nu\\[1em]\nabla_\mu m_\nu=\overline{\nu}\ell_\mu\ell_\nu-\tau\ell_\mu n_\nu+(\gamma-\overline{\gamma})\ell_\mu m_\nu+\overline{\pi}n_\mu\ell_\nu-\kappa n_\mu n_\nu+\\\qquad+(\varepsilon-\overline{\varepsilon})n_\mu m_\nu-\overline{\mu}m_\mu\ell_\nu+\rho m_\mu n_\nu+(\overline{\beta}-\alpha)m_\mu m_\nu-\\\qquad-\overline{\lambda}\overline{m}_\mu\ell_\nu+\sigma\overline{m}_\mu n_\nu+(\overline{\alpha}-\beta)\overline{m}_\mu m_\nu\end{array}\tag{6.13}$$

The field equations (6.5) become

$$\begin{array}{l}m^\mu\partial_\mu\phi_0+\overline{m}^\mu\partial_\mu\overline{\phi_0}+\phi_0[3\beta-2\tau+\overline{\alpha}]+\overline{\phi_0}[3\overline{\beta}-2\overline{\tau}+\alpha]+\\\qquad+2(nmF_j)(n\overline{m}F_j)=0\\[1em]\ell^\mu\partial_\mu\phi_0+\phi_0[3\varepsilon+\overline{\varepsilon}-\rho]+\phi_1[\tau+\overline{\pi}]+(\ell nF_j)(n\overline{m}F_j)=0\end{array}\tag{6.14}$$

The field equations (6.7) become

$$\begin{array}{l}\overline{m}^\mu\partial_\mu\phi_{\tilde{0}}+m^\mu\partial_\mu\overline{\phi_{\tilde{0}}}+\phi_{\tilde{0}}[-3\alpha+2\pi-\overline{\beta}]+\overline{\phi_{\tilde{0}}}[-3\overline{\alpha}+2\overline{\pi}-\beta]-\\\qquad-2(\ell mF_j)(\ell\overline{m}F_j)=0\\[1em]n^\mu\partial_\mu\phi_{\tilde{0}}+\phi_{\tilde{0}}[-3\gamma-\overline{\gamma}+\mu]-\phi_{\tilde{1}}[\overline{\tau}+\pi]-(\ell nF_j)(\ell mF_j)=0\end{array}\tag{6.15}$$

The field equations (6.9) become

$$\begin{array}{l}m^\mu\partial_\mu\overline{\phi_{\tilde{1}}}+\overline{\phi_{\tilde{1}}}[-3\overline{\alpha}+\beta+\overline{\pi}]+\overline{\phi_{\tilde{0}}}[\mu-\overline{\mu}]-(\ell\overline{m}F_j)(m\overline{m}F_j)=0\\[1em]m^\mu\partial_\mu\phi_1+\phi_1[3\beta-\overline{\alpha}-\tau]+\phi_0[\rho-\overline{\rho}]-(n\overline{m}F_j)(m\overline{m}F_j)=0\\[1em]\ell^\mu\partial_\mu\phi_1+n^\mu\partial_\mu\overline{\phi_{\tilde{1}}}+\phi_1[3\varepsilon-2\rho-\overline{\varepsilon}]+\overline{\phi_{\tilde{1}}}[-3\overline{\gamma}+2\overline{\mu}+\gamma]-\\\qquad-2(\ell\overline{m}F_j)(n\overline{m}F_j)=0\end{array}\tag{6.16}$$

Using the Newman-Penrose spin coefficients, the field equations (6.2) become

$$\begin{aligned}
&m^\mu D_\mu(\ell\overline{m}F_j) + \overline{m}^\mu D_\mu(\ell m F_j) + (\ell\overline{m}F_j)[\overline{\pi} - 2\overline{\alpha}]+ \\
&\quad +(\ell m F_j)[\pi - 2\alpha] = 0 \\[6pt]
&m^\mu D_\mu(n\overline{m}F_j) + \overline{m}^\mu D_\mu(n m F_j) + (n\overline{m}F_j)[2\beta - \tau]+ \\
&\quad +(n m F_j)[2\overline{\beta} - \overline{\tau}] = 0 \\[6pt]
&\ell^\mu D_\mu(n m F_j) + n^\mu D_\mu(\ell m F_j) + (n m F_j)[2\overline{\varepsilon} - \overline{\rho}]]+ \\
&\quad +(\ell m F_j)[\mu - 2\gamma] = 0
\end{aligned} \tag{6.17}$$

Their integrability conditions are satisfied identically.

The integrability condition of the equations (6.15) is

$$\begin{aligned}
&m^\mu\partial_\mu[(\ell n F_j)(n\overline{m}F_j)] + \overline{m}^\mu\partial_\mu[(\ell n F_j)(n m F_j)] - 2\ell^\mu\partial_\mu[(n m F_j)(n\overline{m}F_j)]+ \\
&\quad +(2\beta + \overline{\pi} - 2\tau)(\ell n F_j)(n\overline{m}F_j) + (2\overline{\beta} + \pi - 2\overline{\tau})(\ell n F_j)(n m F_j)+ \\
&\quad +(\tau + \overline{\pi})(m\overline{m}F_j)(n\overline{m}F_j) - (\overline{\tau} + \pi)(m\overline{m}F_j)(n m F_j)+ \\
&\quad +2(\rho + \overline{\rho} - 2\varepsilon - 2\overline{\varepsilon})(n m F_j)(n\overline{m}F_j) = 0
\end{aligned} \tag{6.18}$$

where the tetrad commutation relations[6] are used. The equations (6.16) imply

$$\begin{aligned}
&m^\mu\partial_\mu[(\ell n F_j)(\ell\overline{m}F_j)] + \overline{m}^\mu\partial_\mu[(\ell n F_j)(\ell m F_j)] - 2n^\mu\partial_\mu[(\ell m F_j)(\ell\overline{m}F_j)]+ \\
&\quad +(-2\overline{\alpha} + 2\overline{\pi} - \tau)(\ell n F_j)(\ell\overline{m}F_j) + (-2\alpha + 2\pi - \overline{\tau})(\ell n F_j)(\ell m F_j)+ \\
&\quad +(\tau + \overline{\pi})(m\overline{m}F_j)(\ell\overline{m}F_j) - (\overline{\tau} + \pi)(m\overline{m}F_j)(\ell m F_j)+ \\
&\quad +2(2\gamma + 2\overline{\gamma} - \mu - \overline{\mu})(\ell m F_j)(\ell\overline{m}F_j) = 0
\end{aligned} \tag{6.19}$$

and the equations (6.17) imply the integrability condition

$$\begin{aligned}
&\ell^\mu\partial_\mu[(n\overline{m}F_j)(m\overline{m}F_j)] + n^\mu\partial_\mu[(\ell\overline{m}F_j)(m\overline{m}F_j)] - 2m^\mu\partial_\mu[(\ell\overline{m}F_j)(n\overline{m}F_j)]+ \\
&\quad +(2\varepsilon - 2\rho - \overline{\rho})(n\overline{m}F_j)(m\overline{m}F_j) + (2\overline{\mu} - 2\overline{\gamma} + \mu)(\ell\overline{m}F_j)(m\overline{m}F_j)+ \\
&\quad +(\rho - \overline{\rho})(\ell n F_j)(n\overline{m}F_j) + (\overline{\mu} - \mu)(\ell n F_j)(\ell\overline{m}F_j)+ \\
&\quad +2(2\overline{\alpha} - 2\beta + \tau - \overline{\pi})(\ell\overline{m}F_j)(n\overline{m}F_j) = 0
\end{aligned} \tag{6.20}$$

Notice that the curvature terms cancel out in all these integrability conditions.

The above integrability conditions are the null tetrad forms of the following relations implied by the gauge field equations (6.1).

$$\begin{aligned}
&\nabla_\mu\{\Gamma^{\mu\lambda\rho\sigma}F_{j\nu\lambda}F_{j\rho\sigma} - \tfrac{1}{4}\delta^\mu{}_\nu(\Gamma^{\tau\lambda\rho\sigma}F_{j\tau\lambda}F_{j\rho\sigma})\} = -\tfrac{1}{4}(\nabla_\nu\Gamma^{\tau\lambda\rho\sigma})F_{j\tau\lambda}F_{j\rho\sigma} \\[6pt]
&\Gamma^{\mu\nu\rho\sigma} := \tfrac{1}{2}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho\overline{m}^\sigma - n^\sigma\overline{m}^\rho) + (n^\mu\overline{m}^\nu - n^\nu\overline{m}^\mu)(\ell^\rho m^\sigma - \ell^\sigma m^\rho) + c.c.]
\end{aligned} \tag{6.21}$$

which is not exactly a covariant energy conservation form.

7 PATH-INTEGRAL QUANTIZATION

The path-integral quantization can also be accomplished by simply following the ordinary steps. We first see that the local symmetries of the complete action are the usual gauge symmetry, reparametrization and the tetrad-Weyl transformations. For every local symmetry we have to assume a gauge condition. Here we must be careful to impose convenient gauge conditions such that the induced Faddeev-Popov determinant to have vanishing the upper diagonal elements in order to be reduced down into the product of the three determinants, which correspond the three local symmetries of the action. I assume the convenient gauge condition for the usual gauge field local symmetry. The additional tetrad-Weyl symmetry of the tetrad is fixed using the following conditions

$$\begin{array}{ll} \ell^\mu N_\mu - 1 = 0 \quad , & n^\mu L_\mu - 1 = 0 \\ \overline{m}^\mu M_\mu + 1 = 0 \quad , & m^\mu \overline{M}_\mu + 1 = 0 \end{array} \tag{7.1}$$

The convenient conditions which fix the reparametrization symmetry are

$$\begin{array}{ll} L^\mu \ell_\mu n^\nu L_\nu = 0 \quad , & N^\mu n_\mu \ell^\nu N_\nu = 0 \\ M^\mu m_\mu \overline{m}^\nu M_\nu = 0 \quad , & \overline{M}^\mu \overline{m}_\mu m^\nu \overline{M}_\nu = 0 \end{array} \tag{7.2}$$

where $L, N, M, \overline{M}$ is an external (flat) light-cone tetrad, which will be fixed below. Then the Faddeev-Popov terms of the effective lagrangian are the following

$$\begin{array}{l} I_{FP} = \int d^4x \{ -\frac{1}{2\alpha}[\eta^{\mu\nu}\partial_\mu A_{j\nu}]^2 + B_1(\ell^\mu N_\mu - 1) + \\ \quad +B_2(n^\mu L_\mu - 1) + B_3(\overline{m}^\mu M_\mu + 1) + B_4(m^\mu \overline{M}_\mu + 1) + \\ \quad +B_5(L^\mu \ell_\mu)(n^\nu L_\nu) + B_6(N^\mu n_\mu)(\ell^\nu N_\nu) + \\ \quad +B_7(M^\mu m_\mu)(\overline{m}^\nu M_\nu) + B_8(\overline{M}^\mu \overline{m}_\mu)(m^\nu \overline{M}_\nu) + \\ \quad +\eta^{\mu\nu}(\partial_\mu \overline{d}_j)(\partial_\nu d_j - \gamma f_{jik} d_i A_{k\nu}) - \\ \quad -\overline{c}_1 L^\mu[b^\nu(\partial_\nu \ell_\mu) + \ell_\nu(\partial_\mu b^\nu)] - \overline{c}_2 N^\mu[b^\nu(\partial_\nu n_\mu) + n_\nu(\partial_\mu b^\nu)] - \\ \quad -\overline{c}_3 M^\mu[b^\nu(\partial_\nu m_\mu) + m_\nu(\partial_\mu b^\nu)] - \overline{c}_4 \overline{M}^\mu[b^\nu(\partial_\nu \overline{m}_\mu) + \overline{m}_\nu(\partial_\mu b^\nu)]\} \end{array} \tag{7.3}$$

where $\overline{d}_j$ and d_j are the ghost fields, which correspond to the gauge field condition and $\overline{c}_i$, b^μ are the ghost fields which correspond to the reparametrization symmetry. The tetrad-Weyl symmetry on the tetrad does not generate any ghost field.

The BRS transformation of the fields are found by simply replacing the reparametrization parameter, the independent Weyl parameters on the tetrad and the gauge parameters with λb^μ , λc_a and λd_j respectively. We precisely have

7.1 Gauge field propagator in the Landau and Feynman gauges

It is well known that the Landau and Feynman gauges are introduced in the path-integral quantization through a term $\frac{1}{2\alpha}(\eta^{\mu\nu}\partial_\mu A_{j\nu})^2$ in the effective action. The choices $\alpha = 1$ or $\alpha = 0$ are referred as Feynman and Landau gauges respectively. Following the well known path integral technique, the gauge field propagator (for arbitrary α) is

$$\langle T A_{i\mu}(x) A_{j\nu}(y) \rangle = -i\delta_{ij} \int \frac{d^4k}{(2\pi)^4} e^{ik(y-x)} \Delta_{\mu\nu}(k) \tag{7.4}$$

where $\Delta_{\mu\nu}(k)$ satisfies the relation

$$[(L^\rho M^\mu - L^\mu M^\rho)(N^\lambda \overline{M}^\nu - N^\nu \overline{M}^\lambda) + (N^\rho \overline{M}^\mu - N^\mu \overline{M}^\rho)(L^\lambda M^\nu - L^\nu M^\lambda) +$$
$$+ (L^\rho \overline{M}^\mu - L^\mu \overline{M}^\rho)(N^\lambda M^\nu - N^\nu M^\lambda) + (N^\rho M^\mu - N^\mu M^\rho)(L^\lambda \overline{M}^\nu - L^\nu \overline{M}^\lambda) -$$
$$- \tfrac{1}{\alpha}\eta^{\rho\mu}\eta^{\lambda\nu}] k_\rho k_\lambda \Delta_{\mu\nu}(k) = -\delta^\mu_\sigma \tag{7.5}$$

In the present section I will expand the action around the light-cone null tetrad

$$\begin{aligned} E^0_\mu &\equiv L_\mu = \tfrac{1}{\sqrt{2}}(1,\ -1,\ 0,\ 0) \\ E^{\tilde{0}}_\mu &\equiv N_\mu = \tfrac{1}{\sqrt{2}}(1,\ 1,\ 0,\ 0) \\ E^1_\mu &\equiv M_\mu = \tfrac{1}{\sqrt{2}}(0,\ 0,\ 1,\ i) \\ E^{\tilde{1}}_\mu &\equiv \overline{M}_\mu = \tfrac{1}{\sqrt{2}}(0,\ 0,\ 1,\ -i) \end{aligned} \tag{7.6}$$

because the calculations are highly simplified.

Expanding $\Delta_{\nu\sigma}(k)$ in the null tetrad

$$\begin{aligned} \Delta_{\nu\sigma} = {} & H_{00} L_\nu L_\sigma + H_{01}(L_\nu N_\sigma + L_\sigma N_\nu) + H_{02}(L_\nu M_\sigma + L_\sigma M_\nu) + \\ & + \overline{H}_{02}(L_\nu \overline{M}_\sigma + L_\sigma \overline{M}_\nu) + H_{11} N_\nu N_\sigma + H_{12}(N_\nu M_\sigma + N_\sigma M_\nu) + \\ & + \overline{H}_{12}(N_\nu \overline{M}_\sigma + N_\sigma \overline{M}_\nu) + H_{22} M_\nu M_\sigma + \\ & + H_{23}(M_\nu \overline{M}_\sigma + M_\sigma \overline{M}_\nu) + \overline{H}_{22} \overline{M}_\nu \overline{M}_\sigma \end{aligned} \tag{7.7}$$

and substituting into the above relation, a system of linear equations is derived, which can be directly solved. The final result is

$$\begin{aligned} H_{00} &= \tfrac{(Nk)(Nk)}{2(Mk)(\overline{M}k)k^2} + \tfrac{(\alpha-1)(Nk)(Nk)}{k^4} \\ H_{01} &= \tfrac{1}{k^2}\left[1 - \tfrac{(Lk)(Nk)}{2(Mk)(\overline{M}k)k^2} + \tfrac{(\alpha-1)(Lk)(Nk)}{k^2}\right] \\ H_{02} &= \tfrac{(1-\alpha)(Nk)(\overline{M}k)}{k^4} \\ H_{11} &= \tfrac{(Lk)(Lk)}{2(Mk)(\overline{M}k)k^2} + \tfrac{(\alpha-1)(Lk)(Lk)}{k^4} \\ H_{12} &= \tfrac{(1-\alpha)(Lk)(\overline{M}k)}{k^4} \\ H_{22} &= -\tfrac{(\overline{M}k)(\overline{M}k)}{2(Lk)(Nk)k^2} + \tfrac{(\alpha-1)(\overline{M}k)(\overline{M}k)}{k^4} \\ H_{23} &= \tfrac{1}{k^2}\left[-1 + \tfrac{(Mk)(\overline{M}k)}{2(Lk)(Nk)} + \tfrac{(\alpha-1)(Mk)(\overline{M}k)}{k^4}\right] \end{aligned} \tag{7.8}$$

where I denote $(E_a k) \equiv E^\mu_a k_\mu$. These are the light-cone coordinates of the four-vector k_μ. This light-cone notation will be used through out this section in order to keep track of the initial null tetrad structure of the different lagrangian terms.

In the Landau gauge ($\alpha = 0$) the Fourier transform of the gauge field propagator takes the form

$$\begin{aligned} \langle T A_{i\mu}(x) A_{j\nu}(y)\rangle_F = {} & -\tfrac{i\delta_{ij}}{k^2}[\eta_{\mu\nu} - \tfrac{k_\mu k_\nu}{k^2} + \tfrac{(Nk)(Nk)}{2(Mk)(\overline{M}k)} L_\mu L_\nu + \\ & + \tfrac{(Lk)(Lk)}{2(Mk)(\overline{M}k)} N_\mu N_\nu - \tfrac{(Lk)(Nk)}{2(Mk)(\overline{M}k)}(L_\mu N_\nu + L_\nu N_\mu) - \tfrac{(\overline{M}k)(\overline{M}k)}{2(Lk)(Nk)} M_\mu M_\nu + \\ & + \tfrac{(Mk)(\overline{M}k)}{2(Lk)(Nk)}(M_\mu \overline{M}_\nu + M_\nu \overline{M}_\mu) - \tfrac{(Mk)(Mk)}{2(Lk)(Nk)} \overline{M}_\mu \overline{M}_\nu] \end{aligned} \tag{7.9}$$

In the Feynman gauge ($\alpha = 1$) only the ordinary part of the propagator changes to the well known form. The additional non-conventional terms remain the same.

7.2 An appropriate gauge condition

In the Landau and Feynman gauges, the gauge field propagators are very complicated. Therefore they are not convenient for the computation of the Feynman diagrams. I found that the most convenient gauge condition is

$$M^\mu \partial_\mu (\overline{M} A_j) + \overline{M}^\mu \partial_\mu (M A_j) = 0 \tag{7.10}$$

where $(E^a A_j) \equiv E^{a\mu} A_{j\mu}$ are the light-cone coordinates of the gauge field $A_{j\mu}$. I have already used this light-cone notation in the previous subsection.

In the path integral formulation, the validity of a gauge condition is formally assured through the non-annihilation of the Faddeev-Popov determinant. I will check it below in the case of an abelian $U(1)$ gauge field. It is generally assumed that the same results are perturbatively extended to the non-Abelian cases modulo possible Gribov ambiguities. The above gauge condition yields the following Faddeev-Popov operator

$$M_{FP} = -\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \cdot \tag{7.11}$$

The determinant of this operator does not vanish, because it has no regular asymptotically vanishing eigenfunction with zero eigenvalue. One can see it by simply writing this operator in polar coordinates and making a Fourier expansion. Then we see that the zero modes must satisfy the following differential equation

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{n^2}{\rho^2}\right) \Lambda_n(t, x, \rho) = 0 \tag{7.12}$$

For $n \neq 0$ the general solution of this equation is

$$\Lambda_n(t, x, \rho) = h_{1n}(t, x)\rho^n + h_{2n}(t, x)\rho^{-n} \tag{7.13}$$

which is regular at $\rho = 0$ if $h_2 = 0$ and it vanishes at infinity if $h_1 = 0$. For $n = 0$ the solution is

$$\Lambda_0(t, x, \rho) = h_{10}(t, x) + h_{20}(t, x) \ln \rho \tag{7.14}$$

which does not satisfy the regularity conditions. Hence we see that the kernel of the Faddeev-Popov operator contains only the zero function.

One should not be confused by the apparent permitted gauge transformation

$$A'_\mu = A_\mu - \partial_\mu \Lambda(t, x) \tag{7.15}$$

because the asymptotic annihilation is assumed in all space directions. $\Lambda(t, x)$ must vanish because at ρ-infinity it is the same function. Recall that the same argument is applied to the case of the axial gauge condition.

In the conventional procedure, the non-vanishing of the Faddeev-Popov determinant means that the gauge condition uniquely fixes the gauge freedom of the action. The additional point, one should clarify, is that the precise gauge can always be reached starting from any regular asymptotically vanishing field configuration $A_\mu(x)$. This is possible if there is a regular asymptotically vanishing solution to the differential equation

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Lambda = M^\mu\partial_\mu(\overline{M}A_j)+\overline{M}^\mu\partial_\mu(MA_j)\equiv f(x) \tag{7.16}$$

In polar coordinates and after a Fourier expansion it becomes the following ordinary differential equation

$$\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{n^2}{\rho^2}\right)\Lambda_n(t,x,\rho) = f_n(t,x,\rho) \tag{7.17}$$

which always admits a solution with initial conditions

$$\Lambda_n(t,x,0)=0 \quad , \quad \frac{\partial\Lambda_n}{\partial\rho}(t,x,0)=0 \tag{7.18}$$

The above analysis of the convenient gauge condition shows that it is well defined and it may be used to determine the gauge field propagator.

7.3 Lagrangian expansion and propagators

In order to compute the Feynman diagrams we have first to expand the action I_I around a classical solution of the field equations. It is generally believed that the renormalization does not depend on the precise classical solution, but as far as I know, there is no explicit proof of this assumption. In the present case it is convenient to expand around the trivial light-cone tetrad E_a^μ that we have chosen to introduce the conditions which fix the reparametrization and tetrad-Weyl symmetries. That is, we consider the expansion

$$\begin{aligned}
\ell^\mu &= L^\mu + \gamma\varepsilon^\mu_{\tilde 0}\\
n^\mu &= N^\mu + \gamma\varepsilon^\mu_{0}\\
m^\mu &= M^\mu - \gamma\varepsilon^\mu_{\tilde 1}
\end{aligned} \tag{7.19}$$

where γ is a dimensionless constant. Notice that in the lagrangian there is no dimensional constant, which could generate non-renormalizable counterterms through the regularization procedure. In this tetrad expansion, the conditions become

$$\begin{aligned}
&\varepsilon^\mu_{\tilde 0}N_\mu = 0 \quad , \quad \varepsilon^\mu_0 L_\mu = 0 \quad , \quad \varepsilon^\mu_1 M_\mu = 0\\
&\varepsilon^\mu_{\tilde 0}L_\mu - \gamma[(\varepsilon^\nu_{\tilde 0}M_\nu)(\varepsilon^\rho_1 L_\rho)+(\varepsilon^\nu_{\tilde 0}\overline{M}_\nu)(\varepsilon^\rho_{\tilde 1}L_\rho)]+O(\gamma^2)=0\\
&\varepsilon^\mu_0 N_\mu - \gamma[(\varepsilon^\nu_0 M_\nu)(\varepsilon^\rho_1 N_\rho)+(\varepsilon^\nu_0\overline{M}_\nu)(\varepsilon^\rho_{\tilde 1}N_\rho)]+O(\gamma^2)=0\\
&\varepsilon^\mu_1\overline{M}_\mu - \gamma[(\varepsilon^\nu_1 L_\nu)(\varepsilon^\rho_0\overline{M}_\rho)+(\varepsilon^\nu_1 N_\nu)(\varepsilon^\rho_{\tilde 0}\overline{M}_\rho)]+O(\gamma^2)=0
\end{aligned} \tag{7.20}$$

They can be solved and replaced back into the action, which is so expanded in the dimensionless coupling constants γ and q. The first terms of this expansion of the I_I part of the action are the following

$$\begin{array}{l} I_I \simeq \int d^4x\{[(LM\partial A_j)(N\overline{M}\partial A_j) + (L\overline{M}\partial A_j)(NM\partial A_j)]- \\ \quad -qf_{jik}[(LA_i)(MA_k)(N\overline{M}\partial A_j) + (NA_i)(\overline{M}A_k)(LM\partial A_j) + c.c]+ \\ \quad +\gamma[(M\varepsilon_{\widetilde{0}})(M\overline{M}\partial A_j)(N\overline{M}\partial A_j) - (L\varepsilon_{\widetilde{1}})(LN\partial A_j)(N\overline{M}\partial A_j)+ \\ \quad +(N\varepsilon_1)(LM\partial A_j)(LN\partial A_j) - (\overline{M}\varepsilon_0)(LM\partial A_j)(M\overline{M}\partial A_j) + c.c]+ \\ \quad +q^2 f_{jik}f_{ji'k'}[(LA_i)(MA_k)(NA_{i'})(\overline{M}A_{k'}) + c.c]\} \end{array} \tag{7.21}$$

The first terms of the I_C part of the action are

$$\begin{array}{l} I_C \simeq \int d^4x\{-[\phi_0 L^\nu\partial_\nu(L\varepsilon_{\widetilde{1}}) + \phi_1 M^\nu\partial_\nu(M\varepsilon_{\widetilde{0}})+ \\ \quad +\phi_{\widetilde{0}}N^\nu\partial_\nu(N\varepsilon_1) + \phi_{\widetilde{1}}\overline{M}^\nu\partial_\nu(\overline{M}\varepsilon_0) + c.c.]- \\ \quad -\gamma[\phi_0(M\varepsilon_{\widetilde{0}})[M^\nu\partial_\nu(L\varepsilon_1) - \overline{M}^\nu\partial_\nu(L\varepsilon_{\widetilde{1}})]+ \\ \quad +\phi_1(L\varepsilon_{\widetilde{1}})[L^\nu\partial_\nu(M\varepsilon_0) - N^\nu\partial_\nu(M\varepsilon_{\widetilde{0}})]+ \\ \quad +\phi_{\widetilde{0}}(\overline{M}\varepsilon_0)[\overline{M}^\nu\partial_\nu(N\varepsilon_{\widetilde{1}}) - M^\nu\partial_\nu(N\varepsilon_1)]+ \\ \quad +\phi_{\widetilde{1}}(N\varepsilon_1)[N^\nu\partial_\nu(\overline{M}\varepsilon_{\widetilde{0}}) - L^\nu\partial_\nu(\overline{M}\varepsilon_0)] + c.c.]\} \end{array} \tag{7.22}$$

The first terms of the I_{FP} part of the action are

$$\begin{array}{l} I_{FP} = \int d^4x\{-2\overline{d}_j M^\mu\overline{M}^\nu(\partial_\mu\partial_\nu d_j) - \overline{c}_1 L^\mu\partial_\mu(Lc)- \\ \quad -\overline{c}_2 N^\mu\partial_\mu(Nc) - \overline{c}_3 M^\mu\partial_\mu(Mc) - \overline{c}_4\overline{M}^\mu\partial_\mu(\overline{M}c)- \\ \quad -qf_{jik}[M^\mu(\partial_\mu\overline{d}_j)d_i(\overline{M}A_k) + \overline{M}^\mu(\partial_\mu\overline{d}_j)d_i(MA_k)]+ \\ \quad +\gamma[\overline{c}_1(L\varepsilon_1)L^\mu\partial_\mu(Mc) + \overline{c}_1(L\varepsilon_{\widetilde{1}})L^\mu\partial_\mu(\overline{M}c)+ \\ \quad +\overline{c}_2(N\varepsilon_1)N^\mu\partial_\mu(Mc) + \overline{c}_2(N\varepsilon_{\widetilde{1}})N^\mu\partial_\mu(\overline{M}c)+ \\ \quad +\overline{c}_3(M\varepsilon_0)N^\mu\partial_\mu(Lc) + \overline{c}_3(M\varepsilon_{\widetilde{0}})M^\mu\partial_\mu(Nc)+ \\ \quad +\overline{c}_4(\overline{M}\varepsilon_0)\overline{M}^\mu\partial_\mu(Lc) + \overline{c}_4(\overline{M}\varepsilon_{\widetilde{0}})\overline{M}^\mu\partial_\mu(Nc)]\} \end{array} \tag{7.23}$$

where the already defined short light-cone notation is used.

The zeroth order terms of this action expansion determine the field propagator. The Fourier transforms of the gauge field propagator has the following convenient form

$$\begin{array}{l} \langle T\varphi_i\varphi_j\rangle_F = \frac{i\delta_{ij}}{4(Lk)(Nk)(Mk)(\overline{M}k)} \\ \langle T(LA_i)(NA_j)\rangle_F = \frac{i\delta_{ij}}{4(Mk)(\overline{M}k)} \\ \langle T(MA_i)(MA_j)\rangle_F = -\frac{i(\overline{M}k)(Mk)\delta_{ij}}{4(Lk)(Nk)(Mk)(\overline{M}k)} \\ \langle T(\overline{M}A_i)(\overline{M}A_j)\rangle_F = -\frac{i(\overline{M}k)(\overline{M}k)\delta_{ij}}{4(Lk)(Nk)(Mk)(\overline{M}k)} \\ \langle T(MA_i)(\overline{M}A_j)\rangle_F = \frac{i\delta_{ij}}{4(Lk)(Nk)} \end{array} \tag{7.24}$$

where I use the defined previously light-cone short notation

$$\begin{array}{l} (Lk) = \frac{k^0-k^1}{\sqrt{2}} \\ (Nk) = \frac{k^0+k^1}{\sqrt{2}} \\ (Mk) = \frac{k^2+ik^3}{\sqrt{2}} \end{array} \tag{7.25}$$

Notice that this propagator is essentially the product of two well known 2-dimensional scalar field propagator

$$D_{L(E)} = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx}}{k^2 + i\varepsilon} = \frac{i}{4\pi} \int \frac{dt}{t} e^{-i(x^2 - i\varepsilon)t} \tag{7.26}$$

where the indices L and E correspond to the signatures $(+,-)$ and $(-,-)$ respectively. This propagator is logarithmically divergent, but the difference $D(x) - D(x_0)$ is apparently finite. One can easily find that the explicit form of the present gauge field propagator is

$$\begin{array}{l}
\langle T\varphi_i(0)\varphi_j(x)\rangle = -i\delta_{ij} D_L(x^0, x^1) D_E(x^2, x^3) \\
\langle T\,(LA_i(0))\,(NA_j(x))\rangle = -i\delta_{ij}\delta(x^0)\delta(x^1) D_E(x^2, x^3) \\
\langle T\,(MA_i(0))\,(MA_j(x))\rangle = i\delta_{ij} D_L(x^0, x^1) M^\mu M^\nu \partial_\mu \partial_\nu D_E(x^2, x^3) \\
\langle T\,(\overline{M}A_i(0))\,(\overline{M}A_j(x))\rangle = i\delta_{ij} D_L(x^0, x^1) \overline{M}^\mu \overline{M}^\nu \partial_\mu \partial_\nu D_E(x^2, x^3) \\
\langle T\,(MA_i(0))\,(\overline{M}A_j(x))\rangle = i\delta_{ij} D_L(x^0, x^1) \delta(x^2)\delta(x^3)
\end{array} \tag{7.27}$$

The Fourier transforms of the other field propagators are

$$\begin{array}{l}
\langle T\phi_0(L\varepsilon_{\tilde{1}})\rangle_F = -\frac{1}{(Lk)} \quad , \quad \langle T\phi_1(M\varepsilon_{\tilde{0}})\rangle_F = -\frac{1}{(Mk)} \\
\langle T\phi_{\tilde{0}}(N\varepsilon_1)\rangle_F = -\frac{1}{(Nk)} \quad , \quad \langle T\phi_{\tilde{1}}(\overline{M}\varepsilon_0)\rangle_F = -\frac{1}{(\overline{M}k)} \\
\langle T\overline{c}_1(Lc)\rangle_F = \frac{1}{(Lk)} \quad , \quad \langle T\overline{c}_2(Nc)\rangle_F = \frac{1}{(Nk)} \\
\langle T\overline{c}_3(Mc)\rangle_F = \frac{1}{(Mk)} \quad , \quad \langle T\overline{c}_4(\overline{M}c)\rangle_F = \frac{1}{(\overline{M}k)} \\
\langle Td_i\overline{d}_j\rangle_F = \frac{i\delta_{ij}}{2(Mk)(\overline{M}k)}
\end{array} \tag{7.28}$$

Notice that there is no tetrad-tetrad propagator. Only ϕ_b-tetrad propagators exist. There is no loop diagram with ϕ_b external lines. The one-particle irreducible (1PI) diagrams of the model do not contain $\phi - \varepsilon$ and $\overline{c} - c$ propagators. This crucial property implies that there is no divergent candidate to renormalize the term I_H of the action. This means that the regularization procedure does not affect the integrability of the complex structure and subsequently the metric independence of the action in a structure coordinates chart.

7.4 Regularization

The expansion around the constant light-cone tetrad separates the 4-dimensional spacetime into two different 2-dimensional spaces, because in the convenient gauge condition all the field propagators become the product of two 2-dimensional propagators or one 2-dimensional propagator and a 2-dimensional delta function. This is the characteristic property of the special gauge condition which is responsible for the finiteness of the loop diagrams computed below. Any loop-integral turns out to become the product of two independent 2-dimensional integrals. Therefore the dimensional regularization must be simultaneously be performed in both 2-dimensional subspaces. It is done by extending the dimension of the $(L_\mu,\ N_\mu)$-subspace into 2ω and the $(M_\mu,\ \overline{M}_\mu)$-subspace into $2\omega'$.

When the dimension of the spacetime changes into $2(\omega+\omega')$ the number of tetrads changes too. Therefore first the substitutions $2(Lk)(Nk)=k^2$ and $2(Mk')(\overline{M}k')=k'^2$ are made in all the integrals, which are then dimensionally regularized. The results are finally contracted with the remaining tetrads using the formula

$$E_a^\mu E_b^\nu \eta_{\mu\nu}=\eta_{ab} \tag{7.29}$$

which does not contain the spacetime dimension. It does appear after the additional contraction with η^{ab}.

The formula of the dimensional regularization, which will be applied are the called "'t Hooft-Veltman conjecture"

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2)^{\beta-1}=0 \qquad \forall\beta=0,1,2,... \tag{7.30}$$

and the following logarithmically divergent 2-dimensional integral

$$\begin{aligned} I_{\rho\nu}=\int \tfrac{d^{2\omega}k}{(2\pi)^{2\omega}}\tfrac{k_\rho k_\nu}{k^2(k-p)^2}=\eta_{\rho\nu}\tfrac{\Gamma(1-\omega)}{2(4\pi)^\omega}\int_0^1 dx[x(1-x)p^2+\mu^2]^{\omega-1}+\\ +p_\rho p_\nu\tfrac{\Gamma(2-\omega)}{(4\pi)^\omega}\int_0^1 dx x^2[x(1-x)p^2+\mu^2]^{\omega-2} \end{aligned} \tag{7.31}$$

where the ordinary mass term μ^2 has been introduced in order to distinguish the ultraviolet from the infrared divergencies. Notice that in the infrared limit ($\mu^2=0$) the annihilation of the tadpole diagram ($\beta=0$ in the 't Hooft-Veltman conjecture) is rederived.

In the present 2-dimensional case ($\omega=1$) the second term of $I_{\rho\nu}$has no ultraviolet divergence, therefore the following integrals, which appear in the calculations, are finite.

$$\begin{aligned} &\int \tfrac{d^2k}{(2\pi)^2}\tfrac{(Lk)}{(Nk)(L\cdot(k-p))(N\cdot(k-p))}=i(Lp)^2\int_0^1 dx\tfrac{x^2}{x(1-x)(-p^2)+\mu^2}\\ &\int \tfrac{d^2k'}{(2\pi)^2}\tfrac{(Mk')}{(\overline{M}k')(M\cdot(k'-p'))(\overline{M}\cdot(k'-p'))}=(Mp)^2\int_0^1 dx\tfrac{x^2}{x(1-x)(p'^2)+\mu^2} \end{aligned} \tag{7.32}$$

where no-primed k,p denote the $(L_\mu,\ N_\mu)$-subspace and the primed k',p' denote the $(M_\mu,\ \overline{M}_\mu)$-subspace components of the 4-momenta k,p. Analogous results are found in the $(N^\mu N^\nu I_{\mu\nu})$ and $(\overline{M}^\mu\overline{M}^\nu I_{\mu\nu})$ contractions.

7.5 First order one-loop diagrams

It has already been stated that there are no loop diagrams with $\phi_a(x)$ external lines. I will study below the three possible cases of one-loop diagrams, which are a) with external tetrads, b) with two external gauge fields and c) with three external gauge fields. I will use the Bogoliubov-Shirkov procedure for the computation of the S-matrix one-loop terms as time-ordered products. Only the main points will be outlined, because it is practically impossible to present all the calculations here.

a) Diagrams with two external tetrads. These diagrams come from the contractions between internal couplings of I_G, I_H and I_{FP} separately. The ghost field contractions give

$$
\begin{aligned}
&[2\ ext.\ tetrads\ from\ I_{FP}] = -\gamma^2 \int d^4y_1 d^4y_2 \{: (L\varepsilon_1(1))(M\varepsilon_0(2)) : \\
&\cdot \langle T\overline{c}_1(1) M^\mu \partial_\mu (L_\nu c^\nu(2))\rangle \langle T L^\mu \partial_\mu (M_\nu c^\nu(1)) \overline{c}_3(2)\rangle + \\
&: (L_\rho \varepsilon^\rho_{\tilde{1}}(1))(\overline{M}_\tau \varepsilon^\tau_{\tilde{0}}(2)) : \langle T\overline{c}_1(1) \overline{M}^\mu \partial_\mu (L_\nu c^\nu(2))\rangle \langle T L^\mu \partial_\mu (\overline{M}_\nu c^\nu(1)) \overline{c}_4(2)\rangle + \\
&: (N_\rho \varepsilon^\rho_1(1))(M_\tau \varepsilon^\tau_{\tilde{0}}(2)) : \langle T\overline{c}_2(1) M^\mu \partial_\mu (N_\nu c^\nu(2))\rangle \langle T N^\mu \partial_\mu (M_\nu c^\nu(1)) \overline{c}_3(2)\rangle + \\
&: (N_\rho \varepsilon^\rho_{\tilde{1}}(1))(\overline{M}_\tau \varepsilon^\tau_{\tilde{0}}(2)) : \langle T\overline{c}_2(1) \overline{M}^\mu \partial_\mu (N_\nu c^\nu(2))\rangle \langle T N^\mu \partial_\mu (\overline{M}_\nu c^\nu(1)) \overline{c}_4(2)\rangle \}
\end{aligned} \tag{7.33}
$$

where : .... : denotes the Wick product and the integration variables $y_1,\ y_2$ are briefly denoted 1 and 2 respectively.

After the substitution of the propagators and some well known changes of variables, it takes the following form

$$
\begin{aligned}
&[2\ ext.\ tetrads\ from\ I_{FP}] = -\gamma^2 \int d^4y_1 d^4y_2 \{: (L_\rho \varepsilon^\rho_1(1))(M_\tau \varepsilon^\tau_0(2)) : \\
&\cdot \left[\int \frac{d^2k}{(2\pi)^2} \frac{(L(p-k))}{(Lk)}\right] \left[\int \frac{d^2k'}{(2\pi)^2} \frac{(M(p-k'))}{(Mk')}\right] + \\
&+ : (L\varepsilon_{\tilde{1}})(\overline{M}\varepsilon_0) : \left[\int \frac{d^2k}{(2\pi)^2} \frac{(L(p-k))}{(Lk)}\right] \left[\int \frac{d^2k'}{(2\pi)^2} \frac{(\overline{M}(p-k'))}{(\overline{M}k')}\right] + \\
&+ : (N\varepsilon_1)(M\varepsilon_{\tilde{0}}) : \left[\int \frac{d^2k}{(2\pi)^2} \frac{(N(p-k))}{(Nk)}\right] \left[\int \frac{d^2k'}{(2\pi)^2} \frac{(M(p-k'))}{(Mk')}\right] + \\
&+ : (N\varepsilon_{\tilde{1}})(\overline{M}\varepsilon_{\tilde{0}}) : \left[\int \frac{d^2k}{(2\pi)^2} \frac{(N(p-k))}{(Nk)}\right] \left[\int \frac{d^2k'}{(2\pi)^2} \frac{(\overline{M}(p-k'))}{(\overline{M}k')}\right] \}
\end{aligned} \tag{7.34}
$$

where the defined above light-cone notation (...) is occasionally used.

Using the formulas of the regularization subsection one can show that all the above integrals vanish in the context of the dimensional regularization.

The integrals generated by the I_C couplings are analogous to the previous ones and I found that they vanish too. The expression is too long to be written down here, therefore I will compute only the diagram with $(L\varepsilon_1)$ $(N\varepsilon_0)$ external lines in order to show how they look like.

$$
\begin{aligned}
&[(L\varepsilon_1)\ (N\varepsilon_0)\ from\ I_C] = -\gamma^2 \int d^4y_1 d^4y_2 : (L\varepsilon_1)(N\varepsilon_0) : \cdot \\
&\cdot \langle T\overline{\phi}_0 N^\mu \partial_\mu (\overline{M}\varepsilon_{\tilde{0}}))\rangle \langle T L^\nu \partial_\nu (\overline{M}\varepsilon_0) \phi_{\tilde{0}}\rangle = \\
&= -\gamma^2 \int d^4y_1 d^4y_2 \{: (L\varepsilon_1)(N\varepsilon_0) : \int \frac{d^4p}{(2\pi)^4} e^{ip(y_2-y_1)} \cdot \\
&\cdot \left[\int \frac{d^2k}{(2\pi)^2} (Nk)(L(p-k))\right] \left[\int \frac{d^2k'}{(2\pi)^2} \frac{1}{(\overline{M}k')(\overline{M}(p-k'))}\right] \}
\end{aligned} \tag{7.35}
$$

which vanishes because of the 't Hooft-Veltman conjecture applied to the k-integration.

The diagrams from the I_G couplings, with gauge field contractions, are

$$
\begin{aligned}
&[2\ ext.\ tetrads\ from\ I_G] = -\frac{\gamma^2}{2} \int d^4y_1 d^4y_2 \{: (\overline{M}\varepsilon_0)(\overline{M}\varepsilon_0) : [\langle T\,(LM\partial A_j)\,(LM\partial A_k)\rangle \cdot \\
&\cdot \langle T\,(M\overline{M}\partial A_j)\,(M\overline{M}\partial A_k)\rangle + \langle T\,(LM\partial A_j)\,(M\overline{M}\partial A_k)\rangle \langle T\,(M\overline{M}\partial A_j)\,(LM\partial A_k)\rangle] - \\
&-2 : (\overline{M}\varepsilon_0)(N\varepsilon_1) : [\langle T\,(LM\partial A_j)\,(LM\partial A_k)\rangle \langle T\,(M\overline{M}\partial A_j)\,(LN\partial A_k)\rangle + \\
&+ \langle T\,(LM\partial A_j)\,(LN\partial A_k)\rangle \langle T\,(M\overline{M}\partial A_j)\,(LM\partial A_k)\rangle] - \\
&-2 : (M\varepsilon_0)(N\varepsilon_{\tilde{0}}) : [\langle T\,(LM\partial A_j)\,(M\overline{M}\partial A_k)\rangle \langle T\,(M\overline{M}\partial A_j)\,(N\overline{M}\partial A_k)\rangle + \\
&+ \langle T\,(LM\partial A_j)\,(N\overline{M}\partial A_k)\rangle \langle T\,(M\overline{M}\partial A_j)\,(M\overline{M}\partial A_k)\rangle] + similar\ terms \}
\end{aligned} \tag{7.36}
$$

This expression is also too long to be written down. I computed all these integrals and I found that they vanish. The conclusion is that there is no counterterm with two external tetrads.

b) Diagrams with two external gauge fields. The number of these diagrams is quite large, but they can be grouped using the following discrete symmetries of the action

$$\begin{array}{lll} a)\ \ell^\mu \Leftrightarrow n^\mu \quad , & \phi_0 \Leftrightarrow \overline{\phi_0} \quad , & \phi_1 \Leftrightarrow \overline{\phi_1} \\ b)\ m^\mu \Leftrightarrow \overline{m}^\mu \quad , & \phi_a \Leftrightarrow \overline{\phi_a} & \forall\ a \end{array} \tag{7.37}$$

The diagrams with $(LA_i)(LA_j)$ external terms give

$$\begin{aligned} [ext(LA_i)(LA_j)] = -\tfrac{\gamma^2}{2} \int d^4y_1 d^4y_2\ f_{j_1 i_1 k_1}\ f_{j_2 i_2 k_2} : (LA_{i_1})(LA_{i_2}) : \cdot \\ \cdot[\langle T\left(MA_{k_1}\right)\left(MA_{k_2}\right)\rangle\langle T\left(N\overline{M}\partial A_{j_1}\right)\left(N\overline{M}\partial A_{j_2}\right)\rangle+ \\ +\langle T\left(MA_{k_1}\right)\left(N\overline{M}\partial A_{j_2}\right)\rangle\langle T\left(N\overline{M}\partial A_{j_1}\right)\left(MA_{k_2}\right)\rangle+ \\ +\langle T\left(MA_{k_1}\right)\left(\overline{M}A_{k_2}\right)\rangle\langle T\left(N\overline{M}\partial A_{j_1}\right)\left(NM\partial A_{j_2}\right)\rangle+ \\ +\langle T\left(MA_{k_1}\right)\left(NM\partial A_{j_2}\right)\rangle\langle T\left(N\overline{M}\partial A_{j_1}\right)\left(\overline{M}A_{k_2}\right)\rangle + c.c.] = \\ \\ = -\tfrac{i\gamma^2 C}{16(4\pi)^2} \int d^4y_1 d^4y_2 \quad \int \tfrac{d^4p}{(2\pi)^4} e^{ip(y_2-y_1)} : (LA_{i_1})(LA_{i_2}) : \cdot \\ \cdot\delta_{i_1 i_2}\ (Np)^2(Mp)^2(\overline{M}p)^2 I_1(p''^2)[2I_2(-p'^2) - I_1(-p'^2)] \end{aligned} \tag{7.38}$$

where $p'^2 = (p^0)^2 - (p^1)^2$, $p''^2 = (p^2)^2 + (p^3)^2$ and I substituted $f_{jik}\ f_{j'ik} = C\delta_{jj'}$. The finite integrals are

$$I_r(k^2) = \int_0^1 dx \frac{x^r}{x(1-x)k^2 + \mu^2} \tag{7.39}$$

The other one-loop diagrams with two external gauge fields are found to be

$$\begin{aligned} & ext[(LA_i)(LA_j)] = -\tfrac{i\gamma^2 C}{16(4\pi)^2} \int d^4y_1 d^4y_2 \quad \int \tfrac{d^4p}{(2\pi)^4} e^{ip(y_2-y_1)} : (LA_{i_1})(NA_{i_2}) : \cdot \\ & \qquad \cdot\delta_{i_1 i_2}\ (Np)^2(Mp)^2(\overline{M}p)^2 I_1(p''^2)[I_2(-p'^2) - I_1(-p'^2)] \\ & ext[(LA_i)(MA_j)] = 0 \\ & ext[(MA_i)(MA_j)] = 0 \\ & ext[(MA_i)(\overline{M}A_j)] = 0 \end{aligned} \tag{7.40}$$

which are finite. On the other hand the one-loop diagrams with internal ghost lines vanish because of the k-integration. Hence my conclusion is that there is no first order one-loop counterterms with two external gauge fields.

c) Diagrams with three external gauge fields. I wrote down all these diagrams with two and three internal gauge fields. Their number is quite large, but they can be grouped using the above discrete symmetry. I investigated these diagrams and I found that they are all finite. This implies that there is no first order coupling constant renormalization, which means that the first term of the function $\beta(\gamma)$ of the renormalization group equation vanishes.

In current terminology, a lagrangian model is called finite if all its transition amplitudes on mass shell are finite without making use of any infinite renormalization either of the field or of the coupling constants. These amplitudes (on mass shell) do not depend on the regularization procedure or the imposed gauge condition, therefore their finiteness should not depend on these two choices either. The general Green functions of a *finite* field theoretical model may diverge, depending on the used gauge conditions. Apparently the existence of a gauge condition which makes the Green functions finite, imply finiteness of the model. This formal reasoning works well in the case of the $N = 4$ supersymmetric Yang-Mills model, which is finite, because the Green functions are finite in the light-cone gauge condition $(LA_j) = 0$. Therefore the fact that in the precise convenient gauge, that we used in the present calculations, we found that the Green functions are finite, permit us to claim that the present model is also finite in the first order approximation. In a different gauge condition (e.g. Landau or Feynman) the Green functions may not be finite but the cross-sections must be finite.

The fundamental property of the present model is its tetrad-Weyl symmetry and the subsequent metric independence of its action. We also saw that there is no loop diagram with ϕ_b external fields. This means that the regularization procedure does not change the term I_C. Hence the regularization procedure will not change the integrability condition and all its consequences. This formal argument implies that no geometric counterterm could be generated by the renormalization procedure. On the other hand the fact that no dimensional constant appears in the expanded action, no counterterm with mass-dimensionality higher than four can emerge. Hence the permitted gauge field counterterms are restricted to the quadratic FF forms. All these arguments suggest that the present model is formally renormalizable. The existence of possible topological anomalies of the tetrad-Weyl symmetry should not be excluded. Such an anomaly could restrict the gauge group to the observed color group $SU(3)$, in complete analogy to the conformal anomaly of the Polyakov action, the 2-dimensional PCFT, which determines the 26-dimensional spacetime.

REMARK: At the end of Part I, It is time to point out that the description of PCFT in this Research eBook followes its historical evolution. It started as an attempt to find a four-dimensional renormalizable generally covariant action in the context of quantum field theory. In the begining, I was sure that quantum theory will prevail to geometry (general relativity). My strugle with my prejudices is evident from my publications ([31], [32], [33], [34], [35], [36], [37], [38], [**?**], [**?**], [39], [40], [41], [**?**], [43]). Step by step and in a hostile proffessional environment, I finally realized that everything is geometry and that quantum theory is implied by a proper mathematical treatment of the generalized functions, which appear in the lorentzian CR-structure.

Part II

MATHEMATICS OF LORENTZIAN CR-STRUCTURES

Synopsis

The lorentzian Cauchy-Riemann (LCR) structure is the new fundamental notion that replaces the lorentzian riemannian structure of general relativity. It is a special integrability condition formulated in the context of the E. Cartan formalism of moving frames. The Newman-Penrose formalism consists the formulation of general relativity in the same Cartan formalism, which we will often use, in order to stress the similarities and essential differences between these two structures. I start first with the intimate relation between the LCR-structure and the generalized functions (Schwartz distributions). If the LCR-structure is realizable, the application of the holomorphic Frobenius theorem, reveals that the ambient complex manifold is the grassmannian space $G(4,2)$ of the lines of $CP(3)$. The LCR-structure conditions permit the definition of a Kaehler metric with the corresponding LCR-manifold being a lagrangian submanifold. The zero gravity LCR-manifold is identified with the characteristic (Shilov) boundary $U(2)$ of the $SU(2,2)$ classical domain. Identifying the affine Poincaré subgroup of $SU(2,2)$ (in the unbounded realization of the classical domain) with the corresponding physical symmetry, opens up the possibility to define the leptonic sector with LCR solitonic configurations related to ruled surfaces of $CP(3)$. The discrete symmetries, spatial and temporal reflections, left and right chiral parts and charge conjugation are defined. On the other hand the background projective structure is going to permit the identification of the electron with the regular "Kerr-Newman" LCR-manifold with the $g = 2$ gyromagnetic ratio already computed by Carter. Recall that in the context of riemannian geometry, this possibility was impossible, because of the essential diffeomorphic naked ring singularity of the Kerr-Newman metric with the electron mass and angular momentum.

8 LCR-MANIFOLDS

It is well known that Einstein, after his discovery that the gravitational interaction is based on the lorentzian riemannian geometry, tried (and failed) to find a geometric origin for the electromagnetic interaction, either through the torsion (with E. Cartan) or through a five dimensional spacetime (with Kaluza-Klein theory). PCFT is based on the LCR-manifold and its action contains a "peculiar" gauge field which will be identified with the gluonic interaction. But we must first understand the essential similarities and differences between the LCR-manifolds and riemannian manifolds.

The typical mathematical problem of LCR-structure is to find a smooth manifold, which can be covered by a well defined atlas of coordinate charts $[U_i]$, where in every chart there is a smooth tetrad $(\ell\ ,\ n\ ,\ m\ ,\ \overline{m})$ with linearly independent two real and a complex vector, such that there are four generally complex (generalized) functions $(z^0, z^1; z^{\widetilde{0}}, z^{\widetilde{1}})$ called LCR-structure coordinates, which satisfy the following homogeneous partial differential equations

$$\begin{array}{l}
\ell^\mu \partial_\mu z^\alpha = 0 \quad , \quad m^\mu \partial_\mu z^\alpha = 0 \\
n^\mu \partial_\mu z^{\widetilde{\beta}} = 0 \quad , \quad \overline{m}^\mu \partial_\mu z^{\widetilde{\beta}} = 0 \\
\\
dz^0 \wedge dz^1 \wedge dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \neq 0
\end{array} \tag{8.1}$$

Solutions exist if there is an analytic extension of the real chart $U_i \subset \mathbb{R}^4 \subset \mathbb{C}^4$ (at least in the one side of $\mathbb{R}^4$), where the following commutation relations are valid

$$\begin{array}{l}
[\ell^\mu \partial_\mu\ ,\ m^\nu \partial_\nu] = h_{\widetilde{0}}^{\widetilde{0}} \ell^\rho \partial_\rho + h_{\widetilde{0}}^{\widetilde{1}} m^\rho \partial_\rho \\
[n^\mu \partial_\mu\ ,\ \overline{m}^\nu \partial_\nu] = h_0^0 n^\rho \partial_\rho + h_0^1 \overline{m}^\rho \partial_\rho
\end{array} \tag{8.2}$$

The above two arguments are equivalent because of the holomorphic Frobenius theorem. The extension to the complex space is imposed by the need to deal with the involution between the real vector ℓ (n) with its complex partner m $(\overline{m})$. In the neighborhoods of points, where $(z^\alpha(x); z^{\widetilde{\beta}}(x))$ have analytic extensions in both sides of $\mathbb{R}^4$, the structure coordinates are analytic (these points are regular points). In the neighborhoods of points, where $(z^\alpha(x); z^{\widetilde{\beta}}(x))$ have analytic extensions only in the one side of $\mathbb{R}^4$, the structure coordinates are Schwartz distributions, with singular support these points. The singular supports constitute the "particles". Therefore we have to find the distributional solutions with singular supports and study their movements.

If we know the tetrad, the method to find the solutions of such a linear system is well known[7]. Let us consider the homogeneous 1st order PDE of $\ell^\mu \partial_\mu z^\alpha = 0$. Its symbol is $\ell^\mu k_\mu = 0$. We first find the integral curves, say, $\frac{dx_\ell^\mu}{d\sigma} = \ell^\mu(x_\ell^\nu(\sigma))$, which here are characteristic curves of the homogeneous 1st order PDE. The values of the solutions $z^\alpha(x)$ are preserved along each curve, because

$$\frac{dx_\ell^\mu}{d\sigma}\partial_\mu z^\alpha(x(\sigma)) = \frac{dz^\alpha(x(\sigma))}{d\sigma} = 0 \tag{8.3}$$

There are two generally complex functions $z^\alpha(x)$, which determine three real independent integral surfaces, because ℓ^μ is real. The intersection of these three independent surfaces determine one characteristic curve. From any point of $\mathbb{R}^4$ passes one characteristic curve. Hence the parameter σ of the characteristic curve with the values of the three independent real functions may be considered as the coordinates of LCR-manifold in a precise chart. These are the "natural coordinates" of the $\ell^\mu\partial_\mu$ real vector. When we "project" these natural coordinates down to the cartesian coordinates i.e. we change coordinates, singularities (caustics) may emerge. These are the well known singularities of the jacobian determinant of the coordinate change.

The same procedure may be used to solve the 1st order PDE $n^\mu\partial_\mu z^{\widetilde{\beta}} = 0$, with the second real vector $n^\mu\partial_\mu$. The corresponding real integral curve $\frac{dx_n^\mu}{d\chi} = n^\mu(x_n^\nu(\chi))$ is the characteristic curve of the PDE, because its symbol is $n^\mu k_\mu = 0$. The two complex solutions $z^{\widetilde{\beta}}(x)$ (and $\overline{z^{\widetilde{\beta}}(x)}$) determine (at most) three independent real functions, which, combined with χ, form another coordinate system of the LCR-manifold and other set of "natural" coordinates.

The degenerate "light-cone" LCR-structure (4.19) does not have any singularity at a finite point of $\mathbb{R}^4$, nor the corresponding integral curves $x_\ell^\mu(t)$ and $x_n^\mu(t)$ have any singular point in $\mathbb{R}^4$. These are geodesics of the Minkowski metric, which have no singularity either. But this conformity does not appear in the case of the simple degenerate example of the "spherical" LCR-structure (4.20) ($r := |\overrightarrow{x}|$).

$$\begin{array}{l}
\ell^\mu\partial_\mu = \partial_0 - (\frac{x^1}{r}\partial_1 + \frac{x^2}{r}\partial_2 + \frac{x^3}{r}\partial_3) \\
n^\mu\partial_\mu = \partial_0 + (\frac{x^1}{r}\partial_1 + \frac{x^2}{r}\partial_2 + \frac{x^3}{r}\partial_3) \\
m^\mu\partial_\mu = [r(x^3+r) - (x^1+ix^2)x^1]\partial_1 + +[ir(x^3+r) - (x^1+ix^2)x^2]\partial_2 - \\
\quad -(x^1+ix^2)(x^3+r)\partial_3 \\
\\
z^0 = x^0 - r = u \quad , \quad z^1 = \frac{x^1+ix^2}{x^3+r} = \tan\frac{\theta}{2}e^{i\varphi} \\
z^{\widetilde{0}} = x^0 + r = v \quad , \quad z^{\widetilde{1}} = \frac{x^1-ix^2}{x^3+r} = \tan\frac{\theta}{2}e^{-i\varphi}
\end{array} \tag{8.4}$$

Recall that it is singular at the negative z-axis. I use spherical coordinates (r, θ, φ) for "convenience", in fact, because I already know that these are among the "natural" coordinates of the two real 4-vectors of the LCR-tetrad. Using t as the parameter σ (χ respectively) of $\ell^\mu\partial_\mu$ and $n^\mu\partial_\mu$, their characteristic curves and jacobians are

$$\begin{array}{l}
x_\ell^\mu(t) = (t,\ (t-u)\sin\theta\cos\varphi,\ (t-u)\sin\theta\sin\varphi,\ (t-u)\cos\theta) \\
dx^0\wedge dx^1\wedge dx^2\wedge dx^3 = r^2\sin\theta du\wedge dt\wedge d\theta\wedge d\varphi \\
\\
x_n^\mu(t) = (t,\ (v-t)\sin\theta\cos\varphi,\ (v-t)\sin\theta\sin\varphi,\ (v-t)\cos\theta) \\
dx^0\wedge dx^1\wedge dx^2\wedge dx^3 = r^2\sin\theta dt\wedge dv\wedge d\theta\wedge d\varphi
\end{array} \tag{8.5}$$

We see that a singularity occurs at $r = 0$ for both congruences. The characteristic lines passing on $t = 0$ from the singularity have $u = 0 = v$ and they follow opposite directions

$$
\begin{aligned}
x^{\mu}_{\ell}(t) &= t(1,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) \\
x^{\mu}_{n}(t) &= t(1,\ -\sin\theta\cos\varphi,\ -\sin\theta\sin\varphi,\ -\cos\theta)
\end{aligned}
\tag{8.6}
$$

distinguished by the angles. The corresponding Minkowski spacetime does not have any singularity problem. Hence we have to reject the "spherical" LCR-structure but not the riemannian structure. We will see that in the case of the "Kerr-Newman" LCR-structure (the electron soliton) is going to happen the opposite, because of the essential naked singularity of the "electron" gravitational dressing.

8.1 Regular and "analytic" coordinates

The structure coordinates $(z^0, z^1; z^{\widetilde{0}}, z^{\widetilde{1}})$ satisfy the differential equations (4.12)

$$
\begin{array}{l}
dz^0 \wedge dz^1 \wedge d\overline{z^0} \wedge d\overline{z^1} = 0 \\
dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \wedge d\overline{z^0} \wedge d\overline{z^1} = 0 \\
dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \wedge d\overline{z^{\widetilde{0}}} \wedge d\overline{z^{\widetilde{1}}} = 0 \\
\\
dz^0 \wedge dz^1 \wedge dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \neq 0
\end{array}
\tag{8.7}
$$

that is, there are (4.13) two real functions ρ_{11} , ρ_{22} and a complex one ρ_{12}, such that

$$
\begin{array}{l}
\rho_{11}(\overline{z^{\alpha}}, z^{\alpha}) = 0 \quad , \quad \rho_{12}\left(\overline{z^{\alpha}}, z^{\widetilde{\alpha}}\right) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\alpha}}) = 0 \\
\frac{\partial \rho_{ij}}{\partial z^b} \neq 0 \neq \frac{\partial \rho_{ij}}{\partial \overline{z^b}}
\end{array}
\tag{8.8}
$$

We see that the forms of these relations is invariant under the transformations

$$
z'^{\alpha} = f^{\alpha}(z^{\beta}) \quad , \quad z'^{\widetilde{\alpha}} = f^{\widetilde{\alpha}}(z^{\widetilde{\beta}})
\tag{8.9}
$$

which are called LCR-transformations. I point out that the general holomorphic transformations $z'^b = f^b(z^c)$ do not preserve the LCR-structure! In a neighborhood of a point p, a LCR-transformation can simplify a smooth structure to the form

$$
\begin{array}{l}
\operatorname{Im} z^0 = \phi_{11}(\overline{z^1}, z^1, \operatorname{Re} z^0) \ , \ \operatorname{Im} z^{\widetilde{0}} = \phi_{22}(\overline{z^{\widetilde{1}}}, z^{\widetilde{1}}, \operatorname{Re} z^{\widetilde{0}}) \ , \ z^{\widetilde{1}} - \overline{z^1} = \phi_{12}(\overline{z^{a}}, z^{\widetilde{\beta}}) \\
\phi_{11}(p) = \phi_{22}(p) = \phi_{12}(p) = 0 \quad , \quad d\phi_{11}(p) = d\phi_{22}(p) = d\phi_{12}(p) = 0
\end{array}
\tag{8.10}
$$

and the corresponding coordinates are called regular LCR-coordinates in the neighborhood of the point p. The proof is simple[1]. We can generally make translation such that the coordinates of the point p are $z^{\beta}=0=z^{\widetilde{\beta}}$. The next step is to expand ρ_{ij} in to powers of $z^{b}$and keep the linear terms. We precisely have

$$\begin{aligned}
\rho_{11}(\overline{z^{\alpha}};z^{\beta}) &= \sum_{\alpha=0}^{1}(a_{\alpha}z^{\alpha}+\overline{a_{\alpha}z^{\alpha}})+O(2)\\
\rho_{22}(\overline{z^{\widetilde{\alpha}}},z^{\widetilde{\beta}}) &= \sum_{\widetilde{\alpha}=0}^{1}(a_{\widetilde{\alpha}}z^{\widetilde{\alpha}}+\overline{a_{\widetilde{\alpha}}z^{\widetilde{\alpha}}})+O(2)\\
\rho_{22}(\overline{z^{\alpha}},z^{\widetilde{\beta}}) &= \sum_{\widetilde{\alpha}=0}^{1}(b_{\widetilde{\alpha}}z^{\widetilde{\alpha}}+\overline{b_{\alpha}z^{\alpha}})+O(2)
\end{aligned} \tag{8.11}$$

Then we make the linear transformations to the new coordinates

$$\begin{array}{ccc}
z'^{0}=a_{\alpha}z^{\alpha} & , & z'^{1}=-b_{\alpha}z^{\alpha}\\
z'^{\widetilde{0}}=a_{\widetilde{\alpha}}z^{\widetilde{\alpha}} & , & z'^{\widetilde{1}}=b_{\widetilde{\alpha}}z^{\widetilde{\alpha}}
\end{array} \tag{8.12}$$

This linear transformation can be inverted, because the embedding is generic. Besides the transformation does not change the order of the $O(2)$ terms.

The LCR-transformations cannot completely remove (annihilate) the functions ϕ_{ij}. But we may exploit the fact that the LCR manifold is a special maximal totally real submanifold of $\mathbb{C}^4$, which may be trivialized at a neighborhood of their real analytic points. This property is the basis of the picture that we will propose for the geometric picture of the universe and the observed elementary particles there in. Therefore it deserves to analyze it in details.

A general 4-dimensional maximal totally real submanifold[1] is determined by four real functions $\rho_{a}(\overline{y^{b}},y^{b})$ vanishing at the submanifold. Under a general holomorphic (analytic) trasformation $z'^{b}=f^{b}(z^{c})$, they take the simple form

$$\begin{aligned}
&\operatorname{Im} y^{a}=\phi_{a}(\operatorname{Re} y^{b})\\
&\phi_{a}(p)=0 \quad , \quad d\phi_{b}(p)=0
\end{aligned} \tag{8.13}$$

If the real functions $\phi_{a}(x^{b})$, $x^{b}\in\mathbb{R}^{b}$ are additionally real analytic (it is the sum of a converging sequence), the coordinates y^{a} can be chosen so that $\operatorname{Im} y^{a}=0$. Hence at the real analytic points of the LCR submanifolds, we can always make a holomorphic transformation $y^{b}=f^{b}(r^{c})$, so that $\frac{r^{a}-\overline{r^{a}}}{2i}=0$, i.e. the real plane $\mathbb{R}^4$ of $\mathbb{C}^4$. The physically interesting case is when $y^{b}=f^{b}(r^{c})$ has singularities in $\mathbb{C}^4$, which are not compact because of Hartog's lemma. The universe is a real LCR manifold of $\mathbb{C}^4$ and an elementary particle is a set of generalized functions with their representatives identified with the particle potential dressings having integrable singularities at its complex trajectory. Therefore I find necessary to review the generalized functions in the next section.

9 GENERALIZED FUNCTIONS

Let me present an argument that makes the generalized functions indispensable for the study of physically interesting phenomena of PCFT. In the general realm of the ambient complex manifold, the LCR-submanifold is defined by relations of the following form (4.13)

$$\begin{gathered}
\rho_{11}(\overline{z^{\alpha}}, z^{\beta}) = 0 \quad , \quad \rho_{12}(\overline{z^{\alpha}}, z^{\widetilde{\beta}}) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\beta}}) = 0 \\
\frac{\partial \rho_{ij}}{\partial z^b} \neq 0 \neq \frac{\partial \rho_{ij}}{\partial \overline{z^b}}
\end{gathered} \tag{9.1}$$

where ρ_{11} , ρ_{22} are real functions and ρ_{12} is a complex function. Notice the particular dependence of ρ_{ij} on the structure coordinates. The form is invariant under the restricted holomorphic transformations $z'^{\beta} = f(z^{\alpha})$ and $z'^{\widetilde{\beta}} = \widetilde{f}(z^{\widetilde{\alpha}})$. This may be viewed as a totally real CR-structure, which achieves the above form via a holomorphic coordinate transformation $z^{\beta} = f(r^a)$ and $z^{\widetilde{\beta}} = \widetilde{f}(r^b)$. Vice-versa, the above form admits a general smooth transformation[1] such that

$$\frac{r^b - \overline{r^b}}{2i} = y^b\left(\frac{r^a + \overline{r^a}}{2}\right) \tag{9.2}$$

But if the real function $y^b(\cdot)$ is real analytic in a neighborhood of the point $r^a = 0$, there is an analytic transformation $r'^b = f^b(r^a)$, such that the totally real CR-structure becomes completely degenerate

$$\frac{r'^b - \overline{r'^b}}{2i} = 0 \tag{9.3}$$

without any dynamics. That is, the real functions $y^b(x)$ must have points or regions of x, which are not real analytic, for the LCR-structure to have some substantial physical content. This implies that the physically interesting LCR-structures are those with distributional structure coordinates $(z^{\alpha}(x); z^{\widetilde{\beta}}(x))$ with singular supports. This characteristic property of the LCR-structures is very important, because it triggers the "derivation" of quantum mechanics as a consequence of the rigged Hilbert space proper treatment of the generalized functions. Therefore it is necessary to review the notions of distributions.

The notion of distribution (a kind of generalized function[15][50]) was first introduced by Schwartz in order to extend the set of "controllable differentiable" functions. In order to define it, we need a set of test functions $\tau(x)$ on which apply the distribution f as a linear functional. A distribution and its derivative are defined as follows

$$\begin{gathered}
< f, \tau > := \int_{\mathbb{R}} f(x)\tau(x)dx \\
< f', \tau > := -\int_{\mathbb{R}} f(x)\tau'(x)dx
\end{gathered} \tag{9.4}$$

The test functions will be denoted with small greek characters and the distributions with latin characters. The first definition implies that the function representative $f(x)$ of the distribution must be locally integrable (continuous in the case dimension one) function. But the definition of its derivative enlarges the representatives with singular functions, which can be written as higher order derivatives of locally integrable functions. That is, a locally integrable singular function and all is derivatives may be considered as well defined distributions (which are no longer locally integrable).

Schwartz introduced two sets of test functions. We will start with the set $\mathcal{D}$ of the smooth functions with compact support (C_c^∞). Therefore the definition of any order derivative of an integrable function is well defined, because the corresponding derivative of a test function is well defined and it continuous to have compact support. The set of the corresponding linear functionals on $\mathcal{D}$, the distributions, is denoted as $\mathcal{D}'$. Typical examples of distributions are the step (Heavyside) distribution $\mathrm{H}(x)$ and the delta (Dirac) distribution $\delta(x)$ defined using the x_+ continuous function as follows

$$x_+ := \{\begin{smallmatrix} x, & x>0 \\ 0, & x<0 \end{smallmatrix}$$

$$\begin{array}{ll} \mathrm{H}(x) := \frac{d}{dx}x_+ = \{\begin{smallmatrix} 1, & x>0 \\ 0, & x<0 \end{smallmatrix} \quad , & \int\limits_{\mathbb{R}} \mathrm{H}(x)\tau(x)dx = \int\limits_0^\infty \tau(x)dx \\ \delta(x) := \frac{d^2}{dx^2}x_+ = \{\begin{smallmatrix} 0, & x>0 \\ 0, & x<0 \end{smallmatrix} \quad , & \int\limits_{\mathbb{R}} \delta(x)\tau(x)dx = \tau(0) \end{array} \tag{9.5}$$

Notice that x_+ does have "proper" derivative at $x = 0$. But viewed as a Schwartz distribution, it has all the derivatives. the equality of two distributions f and g are equal if $< f, \tau > = < g, \tau > \ \ \forall \tau \in \mathcal{D}$. Hence a distribution f "vanishes" in neighborhood N of a point x if $< f, \tau > = 0 \ \forall \tau(x) \in \mathcal{D}$ with support in N. A point x is called essential of a distribution f, if it does not have a neighborhood in which f vanishes. The support of f , denoted $supp(f)$, is the set of its essential points. The singular support of f, denoted $s - supp(f)$, is the set of its essential points, where its function representative is not a smooth function. Hence

$$\begin{array}{l} supp(x_+) = \mathbb{R}_+ \quad , \quad supp(\mathrm{H}(x)) = \mathbb{R}_+ \quad , \quad supp(\delta(x)) = \{0\} \\ s - supp(x_+) = s - supp(\mathrm{H}(x)) = s - supp(\delta(x)) = \{0\} \end{array} \tag{9.6}$$

The logarithmic function $\ln|x|$ is a singular function. It is a distribution, because $\ln|x| = [x(\ln|x| - 1)]'$, that is, it is the 1st derivative of a continuous function. The negative powers of x are neither locally integrable nor continuous at $x = 0$, but they are distributions, because they are higher order derivatives of the locally integrable function $x(\ln|x| - 1)$. Therefore a simple (naive) way to visualize the generalized functions on $\mathbb{R}$ is to consider the continuous functions which do not have derivatives at some points. Every such function is the basis of a "ladder" of Schwartz distributions which are "derivatives" of the base function. We should see Schwartz definition through the integration as a mathematical trick, because the local integral of a continuous function exists.

In the higher dimensional case of $\mathbb{R}^n$ the base functions of the "ladders" are locally integrable functions. A typical example is the electric (and gravitational potential) $\phi(x) = \frac{e}{|\overrightarrow{x}|}$, which is singular at $\overrightarrow{x} = 0$, but it is a well defined distribution, because

$$\int\limits_0^R \frac{e}{r} 4\pi r^2 \tau(r)dr = 4\pi e \int\limits_0^R r\tau(r)dr < \infty \tag{9.7}$$

The electric field strength $\overrightarrow{E}$ is well defined as a distribution, despite the fact that it is not locally integrable, because it is a derivative of $\phi(x)$.

In order to understand the power of the generalized functions to manage with singularities, let us solve the equation $x^k f(x) = 1$, k a positive integer. A general solution is the distribution

$$f(x) = \tfrac{1}{x^k} + \sum_{j=0}^{k-1} c_j \delta^{(j)}(x) \tag{9.8}$$

where c_j are arbitrary constants and $\delta^{(j)}(x)$ denotes the j derivative of the delta function.

The second set of Schwartz test functions $\mathcal{S}$ is the set of smooth (rapid decaying) functions, which vanish at infinity faster than any polynomial. The tempered distributions are those linear functionals which apply on $\mathcal{S}$. Their set is denoted $\mathcal{S}'$. Notice that $\mathcal{D} \subset \mathcal{S}$ and therefore $\mathcal{S}' \subset \mathcal{D}'$.

The Fourier transform of a tempered distribution is defined through the following steps

$$\begin{aligned}
&\widehat{\tau}(k) = (2\pi)^{\frac{-1}{2}} \int e^{-ikx}\tau(x)dx \quad , \quad \tau(x) = (2\pi)^{\frac{-1}{2}} \int e^{ikx}\widehat{\tau}(k)dk \\
&\widehat{f}(k) := (2\pi)^{\frac{-1}{2}} \int e^{-ikx} f(x)dx \quad , \quad f(x) := (2\pi)^{\frac{-1}{2}} \int e^{ikx}\widehat{f}(k)dk \\
&< f,\tau >:= \int f(x)\tau(x)dx = (2\pi)^{\frac{-1}{2}} \int \widehat{f}(-k)\widehat{\tau}(k)dk =< \widehat{f}(-k),\widehat{\tau}(k) >
\end{aligned} \tag{9.9}$$

The Fourier transform is an isomorphism of $\mathcal{S}$. The Fourier transform of a rapid decaying test function is in $\mathcal{S}$, and any function of S is the Fourier transform of a rapid decay function. Hence the Plancherel theorem is applied and therefore the set of the square integrable functions (L^2) is dense in S. This fundamental property makes the tempered distributions the natural framework of the extended Hilbert space (rigged Hilbert space) of quantum field theory.

The Fourier transform permits the analytic extension of $\widehat{\phi}(k)$ for any $\phi(x) \in C_c^\infty$. It is an entire analytic function with the bound

$$|\widehat{\phi}(k)| \leq \tfrac{K_N e^{R|\operatorname{Im} k|}}{(1+|k|)^N}, \ \forall k \in \mathbb{C}, \ \forall N \in \mathbb{N} \tag{9.10}$$

where K_N is a constant which depends on the integer N, and R is the radius of a sphere containing the support of $\phi(x)$. The inverse is also true (Paley-Wienner theorem). Any entire analytic function with the above bound is the Fourier transform of a C_c^∞ function. This theorem is generalized for $\mathcal{S}'$ distributions, with a different bound relation. A distribution $f(x) \in \mathcal{S}'$ has compact support if $\widehat{f}(k)$ has an analytic continuation to an entire analytic function that satisfies

$$|\widehat{f}(k)| \leq K_N(1+|k|)^N e^{R|\operatorname{Im} k|}, \ \forall k \in \mathbb{C}, \ \forall N \in \mathbb{N} \tag{9.11}$$

The difference between the asymptotic limits of the above Fourier transforms is the basis for the emergence of the wavefront singularities, which play fundamental role in quantum field theory.

Another set of test functions is $\mathcal{E} := C^\infty$ i.e. the set of smooth functions. The set of distributions applying on $\mathcal{E}$ is denoted $\mathcal{E}'$. We have $\mathcal{E}' \subset \mathcal{D}'$ because $\mathcal{D} \subset \mathcal{E}$. Precisely $\mathcal{E}'$ are the $\mathcal{D}'$ (and $\mathcal{S}'$) distributions, which have representatives with compact support.

Sato's hyperfunctions[16],[24] may be viewed as distributions (linear functionals) applied to the real analytic functions (C^ω). Their set is denoted $\mathcal{A}$ and the set of hyperfunctions is denoted $\mathcal{A}'$. The apparent subset relation $\mathcal{D} \subset \mathcal{S} \subset \mathcal{E} \subset \mathcal{A}$ for the test functions implies the reversed relation for the corresponding distributions $\mathcal{A}' \subset \mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$.

9.1 Colombeau generalized functions

We consider the following regularization $f_\varepsilon(x)$ of a distribution $f(x) \in \mathcal{D}'$ using a test function $\eta(x) \in \mathcal{D}$ as "mollifier"

$$\begin{aligned}
&f(x) \in \mathcal{D}' \quad , \quad \eta(x) \in \mathcal{D} \quad , \quad \int_{\mathbb{R}} z^k \eta(z) dz = \delta_{k0} \\
&f_\varepsilon(x) = \int_{\mathbb{R}} [\tfrac{1}{\varepsilon}\eta(\tfrac{y-x}{\varepsilon})] f(y) dy = \int_{\mathbb{R}} \eta(z) f(x+\varepsilon z) dz
\end{aligned} \tag{9.12}$$

where the last integral is found after a change of variables $\frac{y-x}{\varepsilon} = z$. The mollifier $\eta(x)$ may also belong to the space S of test functions, which at infinity $|\eta(x)|$ decrease faster than any power of $|x|$, i.e.

$$\lim_{|x|\to\infty} \eta(x) < O(|x|^{-q}), \quad \forall q \in \mathbb{N} \tag{9.13}$$

Assuming a distribution $\widetilde{f}(x)$, which corresponds to a function $f(x) \in C^m$, the regularization tends to the same $f(x)$ because the function (and its corresponding distribution) can be Taylor expanded up to order m.

$$\begin{aligned}
&\widetilde{f}_\varepsilon(x) = \int_{\mathbb{R}} [\tfrac{1}{\varepsilon}\eta(\tfrac{y-x}{\varepsilon})] \widetilde{f}(y) dy = \int_{\mathbb{R}} \eta(z) \widetilde{f}(x+\varepsilon z) dz \simeq \\
&\simeq \widetilde{f}(x) + O(\varepsilon^m) \quad , \quad \forall m \geqslant 0
\end{aligned} \tag{9.14}$$

Notice that the mollifying procedure embeds the $f(x)$, $g(x) \in C^\infty$, and their product (fg) into the mollified functions. We precisely have

$$\begin{aligned}
&[f_\varepsilon(x)][g_\varepsilon(x)] \simeq [f(x) + O(\varepsilon^m)][g(x) + O(\varepsilon^n)] \simeq [f(x)g(x) + O(\varepsilon^{m+n})] \\
&[(fg)_\varepsilon(x)] \simeq [f(x)g(x) + O(\varepsilon^{m+n})]
\end{aligned} \tag{9.15}$$

These two approaches imply that C^∞ functions respect their product up to some negligible terms. The set of the objects $\widetilde{f}_\varepsilon(x)$ is denoted $\mathcal{E}$.

We use the fact that a singular distribution can always be written as the derivative of a continuous function in order to find the form of its molified representative.

$$\begin{aligned} s_\varepsilon(x) &= \int_{\mathbb{R}} [\tfrac{1}{\varepsilon}\eta(\tfrac{y-x}{\varepsilon})] D^n g(y) dy \quad , \quad g(y) \in C \\ s_\varepsilon(x) &= \int_{\mathbb{R}} \eta(z) D_x^n g(x+\varepsilon z) dz = \tfrac{1}{\varepsilon^n} \int_{\mathbb{R}} \eta(z) D_z^n g(x+\varepsilon z) dz = \\ &= (\tfrac{-1}{\varepsilon})^n \int_{\mathbb{R}} D_z^n \eta(z) g(x+\varepsilon z) dz \simeq O(\tfrac{1}{\varepsilon^n}) \end{aligned} \tag{9.16}$$

The embedding of the Heavyside function (the mollified step function) is

$$\mathrm{H}_\varepsilon(x) = \int_{\mathbb{R}} [\tfrac{1}{\varepsilon}\eta(\tfrac{y-x}{\varepsilon})] \mathrm{H}(y) dy = \int_{\mathbb{R}} dz \eta(z) \mathrm{H}(x+\varepsilon z) = \int_{-\frac{x}{\varepsilon}}^{\infty} dz \eta(z) \simeq O(1) \tag{9.17}$$

while we expect

$$\mathrm{H}_\varepsilon(x) = \int_{\mathbb{R}} [\tfrac{1}{\varepsilon}\eta(\tfrac{y-x}{\varepsilon})] D \tfrac{|y|}{2} dy = \tfrac{-1}{2\varepsilon} \int_{\mathbb{R}} dz D_z \eta(z) |x+\varepsilon z|) \simeq O(\tfrac{1}{\varepsilon}) \tag{9.18}$$

The embedding of the Dirac distribution (the mollified delta function) is

$$\delta_\varepsilon(x) = \int_{\mathbb{R}} [\tfrac{1}{\varepsilon}\eta(\tfrac{y-x}{\varepsilon})] \delta(y) dy = \tfrac{1}{\varepsilon}\eta(\tfrac{-x}{\varepsilon}) \simeq O(\tfrac{1}{\varepsilon}) \tag{9.19}$$

while we expect

$$\delta_\varepsilon(x) = \int_{\mathbb{R}} [\tfrac{1}{\varepsilon}\eta(\tfrac{y-x}{\varepsilon})] D^2 \tfrac{|y|}{2} dy = \tfrac{1}{2\varepsilon^2} \int_{\mathbb{R}} dz D_z^2 \eta(z) |x+\varepsilon z| \simeq O(\tfrac{1}{\varepsilon^2}) \tag{9.20}$$

The mollified square of the step function is (by definition)

$$\begin{aligned} &\mathrm{H}_\varepsilon^2(x) = [\int_{-\frac{x}{\varepsilon}}^{\infty} dz \eta(z)]^2 \\ &\lim_{\varepsilon\to 0} \int_{\mathbb{R}} dx [\mathrm{H}_\varepsilon(x)]^2 \phi(x) = \lim_{\varepsilon\to 0} \{ \int_{-\infty}^{0} dx \phi(x) [0 + 0\varepsilon + ...] + \int_{0}^{\infty} dx \phi(x) [1 + 0\varepsilon + ...] \} = \int_{\mathbb{R}} dx \mathrm{H}(x) \phi(x) \\ &\mathrm{H}_\varepsilon^2(x) \approx \mathrm{H}_\varepsilon(x) \end{aligned} \tag{9.21}$$

which means that $[\mathrm{H}_\varepsilon(x)]^2$ is associated with the $\mathrm{H}_\varepsilon(x)$ generalized function. The square of the Dirac function is

$$\begin{aligned} &\delta_\varepsilon^2(x) = [\tfrac{1}{\varepsilon}\eta(\tfrac{-x}{\varepsilon})]^2 \\ &\int_{\mathbb{R}} dx [\delta_\varepsilon(x)]^2 \phi(x) = \int_{\mathbb{R}} dx \phi(x) [\tfrac{1}{\varepsilon}\eta(\tfrac{-x}{\varepsilon})]^2 = \\ &\qquad = \tfrac{1}{\varepsilon} \int_{\mathbb{R}} dz \phi(\varepsilon z) \eta^2(-z) = \tfrac{\phi(0)}{\varepsilon} \int_{\mathbb{R}} dz \eta^2(-z) + ... \simeq O(\tfrac{1}{\varepsilon}) \end{aligned} \tag{9.22}$$

Hence the square of the delta function is a moderate generalized function.

9.2 Wavefront singularities

From now on we will work with generally complex valued test functions defined in higher dimensional spaces $\mathbb{R}^n$ and the distribution will be a sesquilinear functional

$$\begin{aligned} &< f, \tau >:= \int_{\mathbb{R}^n} \overline{f}(x)\tau(x)d^n x \\ &< \partial_\mu f, \tau >:= -\int_{\mathbb{R}^n} \overline{f}(x)\partial_\mu \tau(x)d^n x \end{aligned} \tag{9.23}$$

where $\overline{f}(x)$ is the complex conjugate of $f(x)$. In the Schwartz localization theorem now any distribution is generally a higher partial derivatives of integrable functions

$$\begin{aligned} &L^1(\mathbb{R}^n) := \{g(x) : \int_{\mathbb{R}^n} g(x)d^n x < \infty\} \\ &f(x) = (\prod_{j=1}^{k} \partial_{\mu_j})g(x) \end{aligned} \tag{9.24}$$

These are locally integrable functions (with possible controllable singularities) which decay at infinity fast enough to have a finite integral. The classical electromagnetic potentials are such $L^1(\mathbb{R}^3)$ solutions which "build" the electromagnetic generalized functions.

A **regular point** x of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ has a neighborhood $U(x)$ and a function $f(x) \in C^\infty(\mathbb{R}^n)$ so that

$$< f, \tau >= \int \overline{f}(x)\tau(x)d^n x \quad , \quad \forall \tau(x) \in \mathcal{D}(\mathbb{R}^n) \ : supp(\tau) \subset U(x) \tag{9.25}$$

All the other points of the distribution are called **singular** and their set is called **singular support** of f. If this function representative of the distribution is multiplied with a test function with compact support, we have a distribution with compact support, which enters in the Paley-Wienner theorem. Its Fourier transform can be analytically extended to a holomorphic function.

Recall the difference of the bounds of the Fourier transforms of test functions $\tau(x)$ and distributions $f(x)$ with compact support

$$\begin{aligned} &|\widehat{\tau}(k)| \leq \tfrac{C_N e^{R|\operatorname{Im} k|}}{(1+|k|)^N} \quad , \quad \forall k \in \mathbb{C}^n \\ &|\widehat{f}(k)| \leq C_N(1+|k|)^N e^{R|\operatorname{Im} k|} \quad , \quad \forall k \in \mathbb{C}^n \end{aligned} \tag{9.26}$$

This difference is used to define the $(x, k \neq 0)$ regular and singular points of the cotangent bundle. We first multiply the function representative of a distribution $f(x)$ with a test function $\phi_c(x)$ with compact support a neighborhood of a point y. After we take the Fourier transform of the distribution $\phi_c(x)f(x)$. If

$$|\widehat{\phi_c f}(k)| \leq \tfrac{C_N}{(1+|k|)^N} \quad , \quad \forall k \in N(p) \tag{9.27}$$

where $N(p)$ is a convex cone of p and the point $(y, p \neq 0)$ is a regular point and direction in a cotangent vector bundle. The set of **non-regular** $(y, p \neq 0)$ is the wavefront of the distribution. Notice that a singular point y of a distribution may have "good" and "bad" (wavefront) directions p. The Bogoliubov-Epstein-Glaser[10] work revealed that all the non-renormalizability problems of a quantum field theory are caused by the existence of "bad" (wavefront) directions in the terms of the perturbative expansion of the S-matrix. In order to describe it, we have to describe how the wavefront singularities pass from two distributions to their "product". But product of functionals (Schwartz distributions) does not exist! This is achieved through the Gelfand introduction of the notion of rigged Hilbert space. The tempered distributions are viewed as operators in the rigged Hilbert space $\mathcal{S} \to H \to \mathcal{S}'$, which will be briefly reviewed.

The product of two distributions f_1 and f_2 at a point x is defined after localizing them at x through their multiplication with a smooth function $\phi(x)$ with $supp(\phi) \subset U(x)$, a neighborhood of x, and taking the convolution

$$\widehat{\phi^2 g}(k) = \int \widehat{\phi f_1}(p)\widehat{\phi f_2}(p-k)dp \tag{9.28}$$

The product $g(x) \equiv f_1(x)f_2(x)$ exists if the above convolution integral absolutely converges. Apparently, if the two distributions do not have common singular points, the product exists. But there are possibilities, where the product exists even if they have common singular points. This means that a singularity of a localized distribution needs both $(x, k \neq 0)$ of the Fourier transform variables, which transform as a point of cotangent vector bundle. That is, a singular point x may have "good" and "bad" directions relative to their asymptotic behavior. As a tempered distribution we always have $\widehat{\phi f}(k) \leq C_N(1+|k|)^N,\ \forall k$. But in some cases we have "good" directions p where $\widehat{\phi f}(p) \leq \frac{C_N}{(1+|p|)^N},\ \forall N$, falls off faster than any polynomial, like a regular test function. That is, the Fourier transform of the localized distribution behaves as the Fourier transform of a test function of $\mathcal{S}(\mathbb{R}^n)$. This implies that, if at a point x a direction p is "good" for at least either f_1 or f_2, the above convolution integral exists and the product of the two distributions exists.

There is the following strong criterion[46]: If the set

$$WF(f_1) \oplus WF(f_2) \equiv \{(x, k_1 + k_2) | (x, k_j) \in WF(f_j)\} \tag{9.29}$$

does not contain any element of the form $(x, 0)$, then the product of the two distributions exists and

$$WF(f_1 f_2) \subset WF(f_1) \cup WF(f_2) \cup [WF(f_1) \oplus WF(f_2)] \tag{9.30}$$

Example 2. The wavefront of the delta function in $\mathbb{R}^2$ is

$$\begin{array}{l} WF(\delta(x_1)) = \{(0, x_2; k_1, 0) : x_2 \in \mathbb{R}, k_1 \neq 0\} \\ WF(\delta(x_2)) = \{(x_1, 0; 0, k_2) : x_1 \in \mathbb{R}, k_2 \neq 0\} \\ \\ WF(\delta(x_1)\delta(x_1)) = \{(0; k_1 + k_2) : k_1 \neq 0, k_2 \neq 0\} \\ WF(\delta(x_1)\delta(x_2)) = \{(0, 0; k_1, k_2) : k_1 \neq 0, k_2 \neq 0\} \end{array} \tag{9.31}$$

Notice that the product $\delta(x_1)\delta(x_1)$ does not exist because $(0, k_1 + k_2 = 0) \in WF(\delta(x_1)\delta(x_1))$, while $\delta(x_1)\delta(x_2) \equiv \delta(x_1, x_2)$ does exist.

Another very useful (in quantum field theory) property of the generalized functions is the effect of the elliptic and strictly hyperbolic P operators applied to generalized functions[49]. We generally have

$$WF(f) \subset WF(Pf) \cup Characteristics(P) \tag{9.32}$$

That is a (pseudo-differential) operator diminishes the wavefront of the generalized function, where it applies. That is the solution f of a partial differential equation $Pf = h$ has larger wavefront than h, and precisely by the set of the characteristics of the operator P.

9.3 de Rham currents

In $\mathbb{R}^n$ a current of degree p is a sesquilinear functional

$$\begin{aligned} &< f, \tau >:= \int_{\mathbb{R}^n} \overline{f}(x) \wedge \tau(x) \\ &< dh, \tau >:= -\int_{\mathbb{R}^n} \overline{h}(x) \wedge d\tau(x) d^n x \end{aligned} \tag{9.33}$$

where $\overline{f}(x)$ is the complex conjugate of a p-form $f(x)$, which applies on the $n-p$ test forms $\tau(x)$ with compact support. The second line defines the differential form of a $p-1$ form $h(x)$. Apparently the Schwartz distributions are 0-degree de Rham currents and vice-versa the deRham currents are differential forms with distribution coefficients. Using coordinates and a representative of the p-form $f_{i_1 \dots i_p}$ we find the formula

$$\begin{aligned} &< f, \tau >:= \tfrac{1}{n!} \int_{\mathbb{R}^n} \overline{f_{i_1 \dots i_p}}(x) \tau_{i_{p+1} \dots i_n}(x) \epsilon^{i_1 i_2 \dots i_n} d^n x \\ &< dh, \tau >:= -\tfrac{1}{n!} \int_{\mathbb{R}^n} \overline{\partial_{i_1} h_{i_1 \dots i_p}}(x) \tau_{i_{p+1} \dots i_n}(x) \epsilon^{i_1 i_2 \dots i_n} d^n x \end{aligned} \tag{9.34}$$

A typical example appears in the case of the Coulomb field and the corresponding potential

$$\begin{aligned} &\overrightarrow{E} = \tfrac{q}{4\pi} \tfrac{\overrightarrow{r}}{r^3} \quad \Longleftrightarrow \quad A_0 = \tfrac{-q}{4\pi r} \\ &F \simeq d(A_0 dt) \end{aligned} \tag{9.35}$$

Notice that the field and its potential are singular at $\overrightarrow{x} = 0$. The field F is not a proper derivative of A at the $\mathbb{R}^4$. But the potential A_0 is locally integrable, therefore it can be taken as the base of a ladder of distributions,which may not be locally integrable. Hence, viewed as distributions we can write the equation $F = d(A_0 dt)$.

10 THE AMBIENT COMPLEX MANIFOLD

We have already seen that if a LCR-structure is realizable, it becomes a special totally real submanifold of a complex manifold. The embedding functions (9.1)

$$\begin{aligned}
&\rho_{11}(\overline{z^{\alpha}}, z^{\beta}) = 0 \quad , \quad \rho_{12}(\overline{z^{\alpha}}, z^{\widetilde{\beta}}) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\beta}}) = 0 \\
&\tfrac{\partial \rho_{ij}}{\partial z^{b}} \neq 0 \neq \tfrac{\partial \rho_{ij}}{\partial \overline{z^{b}}}
\end{aligned} \tag{10.1}$$

are in a special (related to the LCR-structure) coordinate patch of the ambient complex manifold. The general coordinate system of this (ambient) complex manifold needs some elementary clarifications. In every LCR-coordinate patch the embedding conditions ρ_{ij} are determined up to non-vanishing factors A_{ij} depending on the same corresponding structure coordinates. We should also be aware that the ambient complex manifold is a mathematical useful notion. It is analogous to the embedding of any riemannian manifold to a higher dimensional flat manifold. It does not exist in nature.

Recall that we first made a complexification of the (real) coordinates x^{μ} of the 4-dimensional real LCR-manifold, imposed by the need to apply the holomorphic Frobenius theorem. This means that we pass to a complex manifold with coordinates $r^{I} = (r^{a}, \overline{r^{b}})$ and the previous real transition functions (in the coordinate patches) become holomorphic. Of course this complexification may generate singularities in the holomorphic transition functions, but we may forget it now, because in practice we will essentially use the inverse procedure. This provides a trivial background complex structure $\widehat{J}$. The integrability conditions of the LCR-structure determine special coordinates $z^{a}(r^{c}) = (z^{\beta}, z^{\widetilde{\beta}})$, via the indicated holomorphic transformations. This defines a new meaningful complex structure J, which apparently commutes with the trivial one $\widehat{J}$. In the complex structure J the complex coordinates are $z^{I} = (z^{\beta}, z^{\widetilde{\beta}}; \overline{z^{\beta}}, \overline{z^{\widetilde{\beta}}})$ with

$$\begin{aligned}
&J(dz^{\beta}) = idz^{\beta} \ , \ J(dz^{\widetilde{\beta}}) = -idz^{\widetilde{\beta}} \\
&J(d\overline{z^{\beta}}) = -id\overline{z^{\beta}} \ , \ J(d\overline{z^{\widetilde{\beta}}}) = id\overline{z^{\widetilde{\beta}}} \\
\\
&\widehat{J}(dz^{\beta}) = idz^{\beta} \ , \widehat{J} \ (dz^{\widetilde{\beta}}) = idz^{\widetilde{\beta}} \\
&\widehat{J}(d\overline{z^{\beta}}) = -id\overline{z^{\beta}} \ , \ \widehat{J}(d\overline{z^{\widetilde{\beta}}}) = -id\overline{z^{\widetilde{\beta}}}
\end{aligned} \tag{10.2}$$

At the intersections of these special patches the coordinates transform as $(z'^{\alpha}, z'^{\widetilde{\alpha}}) = (f^{\alpha}(z^{\beta}), f^{\widetilde{\alpha}}(z^{\widetilde{\beta}}))$, where $f^{\alpha}(z^{\beta})$ and $f^{\widetilde{\alpha}}(z^{\widetilde{\beta}})$ are holomorphic functions. In the context of these two complex structures we may say that the lorentzian CR-structure is a totally real CR-structure restricted to the above transformations in the ambient complex 4-dimensional manifold.

After the first lift of the 4(real)-dimensional LCR-manifold to the 4(complex)-dimensional complex manifold, we will now find a second lift to the hypersurfaces of $CP(3)$. This suggested by the well known Kerr theorem in Minkowski space. This second lift consists to projectivize the LCR-structure conditions (9.1)

$$\begin{aligned}
&\rho_{11}(\overline{Z^{m1}}, Z^{n1}) = 0 \quad , \quad \rho_{12}\left(\overline{Z^{m1}}, Z^{n2}\right) = 0 \quad , \quad \rho_{22}(\overline{Z^{m2}}, Z^{n2}) = 0 \\
&K(Z^{m1}) = 0 = K(Z^{m2})
\end{aligned} \tag{10.3}$$

where $K(Z^n)$ is a homogeneous function function in $\mathbb{C}^4$.

10.1 The grassmannian manifold G(4,2)

The rank two 4×2 complex matrices with equivalence relation

$$X \sim Y \quad if\ \exists\ 2 \times 2\ matrix\ \lambda\ (\det \lambda \neq 0)\ :\ Y = X\lambda \tag{10.4}$$

is the $G(4,2)$ compact complex manifold. The charts of its typical non-homogeneous coordinates are determined by the invertible pairs of rows. If the first two rows constitute an invertible matrix, the chart is determined by $\det Y_1 \neq 0$ and the projective coordinates w are defined by

$$\begin{aligned}
&Y = \begin{pmatrix} Y^{01} & Y^{02} \\ Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \\ Y^{31} & Y^{32} \end{pmatrix} =: \left(\begin{array}{c} Y_1 \\ Y_2 \end{array} \right) = \left(\begin{array}{c} Y_1 \\ wY_1 \end{array} \right) \\
&w = Y_2 Y_1^{-1}
\end{aligned} \tag{10.5}$$

The other five charts of the atlas are analogously defined. The coordinates Y are called homogeneous coordinates and the coordinates w are called projective coordinates. Under a general linear 4×4 transformation

$$\left(\begin{array}{c} Y_1' \\ w'Y_1' \end{array} \right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \left(\begin{array}{c} Y_1 \\ wY_1 \end{array} \right) \tag{10.6}$$

the projective coordinates of the first chart transform as follows

$$w' = (A_{21} + A_{22}\ w)\,(A_{11} + A_{12}\ w)^{-1} \tag{10.7}$$

It is called linear fractional transformation and it is an automorphism of the compact manifold $G_{4,2}$. The chart $\det Y_2 \neq 0$ is the corresponding "infinity" chart with projective coordinates $w' = Y_1 Y_2^{-1}$ and transition function $w' = w^{-1}$.

The grassmannian manifold may be viewed as the lines of $CP(3)$ determined by the two distinct points of the columns of Y. The projectivization of the embedding functions (9.1) of the LCR-structure has the form

$$\begin{aligned}
&\rho_{11}(\overline{Y^{m1}}, Y^{n1}) = 0 \quad , \quad \rho_{12}\left(\overline{Y^{m1}}, Y^{n2}\right) = 0 \quad , \quad \rho_{22}(\overline{Y^{m2}}, Y^{n2}) = 0 \\
&K(Y^{m1}) = 0 = K(Y^{m2})
\end{aligned} \tag{10.8}$$

where $K(Z^m)$ is a homogeneous analytic function, which we will call Kerr function. I call Kerr function the hypersurface analytic homogeneous function $K(Z^m)$, because in the derived general relativity, it will coincide with the analytic function of the Kerr theorem. A line of $CP(3)$ intersects a surface to a number of points equal to the degree d of the surface. On the other hand a line implies a projectivization of $CP(3)$ to a CP^2 subspace with d sheets. Two intersection points of the lines of $CP(3)$ with a hypersurface $K(Z^m) = 0$ of $CP(3)$ determine the structure coordinates of the LCR-structure. That is, the ambient complex manifold of a realizable LCR-structure may be identified with the grassmannian manifold $G(4,2)$. The general $SL(4,\mathbb{C})$ linear transformation of $G_{4,2}$ preserves the form of the LCR-structure embedding conditions. But the LCR-structure solution (10.8) is not covariant with the grassmannian equivalence relation $X \sim X\lambda$ with $\det \lambda \neq 0$. Notice that the structure coordinates z^α is directly related with one sheet of the hypersurface $K(Z^m) = 0$ of $CP(3)$ (the left column of the homogeneous coordinates of $G(4,2)$) and $z^{\widetilde{\beta}}$ with a second sheet of the hypersurface. Hence every pair of sheets i.e. the structure coordinates $(z^\alpha, z^{\widetilde{\beta}})$ and the pairs $[(\ell, m)\ ,\ (n, \overline{m})]$ of the tetrad of the LCR-structure projectively "communicate" through the surface of $CP(3)$.

The advantage of the projectivization is the application of Chow's theorem, that asserts that any analytic subvariety in projective space is an algebraic (polynomial) subvariety. Hence our study of subvarieties may be restricted to the study of polynomials.

10.2 The SU(2,2) symmetric classical domain

Following the Piatetski-Shapiro approach[30], the $SU(2,2)$ symmetric bounded classical domain is the set of points of $G_{4,2}$ with positive definite 2×2 matrix

$$\begin{aligned}
&\begin{pmatrix} Y_1^\dagger & Y_2^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \succ 0 \quad \Longleftrightarrow \quad I - w^\dagger w \succ 0 \\
&w \equiv Y_2 Y_1^{-1}
\end{aligned} \tag{10.9}$$

This is the bounded realization of the $SU(2,2)$ classical domain, which corresponds to the unit disk domain of the plane. The linear transformations which preserve the hermitian matrix have the following form

$$\begin{aligned}
&\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&w' = (A_{21} + A_{22}\ w)\,(A_{11} + A_{12}\ w)^{-1} \\
&A_{11}^\dagger A_{11} - A_{21}^\dagger A_{21} = I \quad , \quad A_{11}^\dagger A_{12} - A_{21}^\dagger A_{22} = 0 \quad , \quad A_{22}^\dagger A_{22} - A_{12}^\dagger A_{12} = I
\end{aligned} \tag{10.10}$$

But if we use instead the following unitary transformation of the hermitian matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \tag{10.11}$$

the classical domain takes the unbounded form

$$\begin{aligned}&\begin{pmatrix}X_1^\dagger & X_2^\dagger\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}\succ 0 \quad\Longleftrightarrow\quad -i(r-r^\dagger)\succ 0\\ &r\equiv iX_2X_1^{-1}\end{aligned}\tag{10.12}$$

where the homogeneous coordinates of the bounded Y and unbounded X realizations are related with the following unitary transformations

$$\begin{aligned}X&=\begin{pmatrix}X_1\\ X_2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}I & I\\ I & -I\end{pmatrix}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}\\ Y&=\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}I & I\\ I & -I\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}\end{aligned}\tag{10.13}$$

The corresponding bounded z and unbounded r projective coordinates transform as follows

$$\begin{aligned}r&=i(I-w)(I+w)^{-1}=i(I+w)^{-1}(I-w)\\ w&=(iI-r)(iI+r)^{-1}=(iI+r)^{-1}(iI-r)\end{aligned}\tag{10.14}$$

It is a Cayley transformation completely analogous to the bounded disk domain and the unbounded upper half-plane domain. That is, the Cartan (bounded) domain and the Siegel (unbounded) domain are viewed as different realizations of the same classical domain.

The general linear transformation of the homogeneous coordinates, which preserves the Siegel (unbounded) domain has the form

$$\begin{aligned}&\begin{pmatrix}X_1'\\ X_2'\end{pmatrix}=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}\\ &r'=\left(B_{22}\,r+iB_{21}\right)\left(B_{11}-iB_{12}\,r\right)^{-1}\\ &B_{11}^\dagger B_{22}+B_{21}^\dagger B_{12}=I \quad,\quad B_{11}^\dagger B_{21}+B_{21}^\dagger B_{11}=0 \quad,\quad B_{22}^\dagger B_{12}+B_{12}^\dagger B_{22}=0\end{aligned}\tag{10.15}$$

where the fractional transformation of the corresponding projective coordinates is also indicated. Notice that if $B_{12}=0$, the transformation becomes an element of the Poincaré×Dilation group, in its spinorial representation

$$\begin{aligned}&\begin{pmatrix}X_1'\\ X_2'\end{pmatrix}=\begin{pmatrix}B & 0\\ -iTB & (B^\dagger)^{-1}\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}\\ &\det B=1 \quad,\quad T^\dagger=T\end{aligned}\tag{10.16}$$

It is important, because we will identify this Poincaré subgroup with the Poincaré group observed in physics.

The smallest (Shilov) boundary of the bounded realization of the $SU(2,2)$ symmetric domain is $w^\dagger w=I$ i.e. the $U(2)$ manifold. The Shilov boundary of the unbounded realization is the "real axis" $y = 0$, where $y := \frac{-i}{2}(r - r^\dagger)$ is the imaginary hermitian matrix of $r =: x + iy$. It is evident that we have to identify the cartesian coordinates of the Minkowski spacetime with

$$\begin{aligned} & x = \eta_{\mu\nu} x^\mu \sigma^\nu = \begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} \\ & x^0 = ct \quad , \quad x^i = (x,\ y,\ z)^\top \end{aligned} \tag{10.17}$$

Notice the appearance of the constant c (with velocity units), which fits the "time" and "length" units. It turns out to become the velocity of light in the vacuum. This identification implies the identification of the present Poincaré group with the physically observed symmetry. Recall that the projectivization permitted the above identifications of the light velocity and the Poincaré group. Hence there must be a fundamental length R_0 for the transformation between the bounded and unbounded realizations to be self-consistent

$$\begin{aligned} & r = iR_0(I - w)(I + w)^{-1} = iR_0(I + w)^{-1}(I - w) \\ & w = (iR_0 I - r)(iR_0 I + r)^{-1} = (iR_0 I + r)^{-1}(iR_0 I - r) \end{aligned} \tag{10.18}$$

In fact we should have already introduced it in the projective coordinates $r \to \frac{1}{R_0} r$ of $G(4,2)$. That is we should have defined $\frac{1}{R_0} r \equiv iX_2 X_1^{-1}$. Its physical meaning is expected to appear.

10.3 Algebraic definition of gravity

We first observe that the defining condition of the Shilov boundary in the unbounded realization

$$\begin{aligned} & \rho_{ij}(\overline{X^{mi}}, X^{nj}) = \overline{X^{mi}} E^U_{mn} X^{nj} = 0 \\ & K(X^{m1}) = 0 = K(X^{m2}) \quad , \quad E^U_{mn} := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \end{aligned} \tag{10.19}$$

has exactly the form of the embedding conditions of the LCR-structure. Hence, for these LCR-structures, which we call "flat", we have real projective coordinates $r = x = x^\dagger$, and the LCR-structure is determined only by the homogeneous holomorphic function $K(X^m)$. That is, the points of the Shilov boundary take the representation

$$\begin{aligned} & X^{mj} = \begin{pmatrix} \lambda \\ -ix\lambda \end{pmatrix} = \begin{pmatrix} \lambda^{Aj} \\ -ix_{A'B}\lambda^{Bj} \end{pmatrix} \\ & x = \eta_{\mu\nu} x^\mu \sigma^\nu = \begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} = x_{A'B} \end{aligned} \tag{10.20}$$

in homogeneous coordinates. Under a Poincaré transformation, they transform as follows

$$
\begin{gathered}
\begin{pmatrix} \lambda' \\ -ix'\lambda' \end{pmatrix} = \begin{pmatrix} B & 0 \\ -iTB & (B^{\dagger})^{-1} \end{pmatrix} \begin{pmatrix} \lambda \\ -ix\lambda \end{pmatrix} \\
\lambda' = B\lambda \quad , \quad x' = (B^{-1})^{\dagger} x B^{-1} + T \\
\det B = 1 \quad , \quad T^{\dagger} = T
\end{gathered}
\tag{10.21}
$$

In order to osculate the general LCR-structure relations with the flat LCR-structure conditions I write

$$
\begin{aligned}
& \rho_{ij}(\overline{X^{mi}}, X^{nj}) = \overline{X^{mi}} X^{nj} E_{mn} - G_{ij}(\overline{X^{mi}}, X^{nj}) = 0 \\
& K(X^{m1}) = 0 = K(X^{m2})
\end{aligned}
\tag{10.22}
$$

Using the following spinorial form of the rank-2 matrix X^{mj} in its unbounded realization

$$
X^{mj} = \begin{pmatrix} \lambda^{Aj} \\ -ir_{A'B}\lambda^{Bj} \end{pmatrix}
\tag{10.23}
$$

and the null tetrad

$$
\begin{aligned}
& L^a = \tfrac{1}{\sqrt{2}} \overline{\lambda}^{A'1} \lambda^{B1} \sigma^a_{A'B} \quad , \quad N^a = \tfrac{1}{\sqrt{2}} \overline{\lambda}^{A'2} \lambda^{B2} \sigma^a_{A'B} \quad , \quad M^a = \tfrac{1}{\sqrt{2}} \overline{\lambda}^{A'2} \lambda^{B1} \sigma^a_{A'B} \\
& \epsilon_{AB} \lambda^{A1} \lambda^{B2} = 1
\end{aligned}
\tag{10.24}
$$

the above relations take the form

$$
\begin{aligned}
& 2\sqrt{2} y^a L_a = G_{11}(\overline{X^{m1}}, X^{n1}) \\
& 2\sqrt{2} y^a \overline{M}_a = G_{12}(\overline{X^{m1}}, X^{n2}) \\
& 2\sqrt{2} y^a N_a = G_{22}(\overline{X^{m2}}, X^{n2})
\end{aligned}
\tag{10.25}
$$

where y^a is the imaginary part of the projective coordinate $r^a = x^a + iy^a$ defined by the relation $r_{A'B} = r^a \sigma_{aA'B}$ and $\sigma^a_{A'B}$ being the identity and the three Pauli matrices. The normalization of the spinors is permitted because of the homogeneity of the functions. These conditions are formally "solved" by

$$
y^a = \tfrac{1}{2\sqrt{2}} [G_{22} N^a + G_{11} L^a - G_{12} M^a - \overline{G_{12} M}^a]
\tag{10.26}
$$

which, combined with the computation of λ^{Ai} as functions of r^a, and using the Kerr condition $K(X^{mi})$, permit us to perturbatively compute y^a as functions of the real part of r^a. This procedure gives the canonical form $y^a = h^a(x)$ of the (totally real) lorentzian CR submanifold expressed in the projective coordinates of $G_{4,2}$. The explicit form of $h^a(x)$ is implied by the precise dependence of $G_{ij}(\overline{X^{mi}}, X^{mj})$ and their expansion into a series relative to h^a.

Notice that this surface does not generally belong into the Seigel domain, because y^0 and

$$y^a y^b \eta_{ab} = \frac{1}{8}[G_{22}G_{11} - G_{12}\overline{G_{12}}] \tag{10.27}$$

are not always positive. But the regular surfaces (with an upper bound) can always be brought inside (or outside) the Siegel domain (and its corresponding holomorphic bounded classical domain) with an holomorphic complex time translation. That is we may take

$$\begin{pmatrix} \lambda' \\ -ir'\lambda' \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ d & \mathbf{1} \end{pmatrix} \begin{pmatrix} \lambda \\ -ir\lambda \end{pmatrix} \tag{10.28}$$

with d a real constant.

It is important to point out that the bounded realization is the patch of the grassmannian space, which permits the global view of the $SU(2,2)$ classical domain and subsequently its boundary, Minkowski spacetime. But the Cayley transformation of the boundary is $\mathbb{R}^4 \ni x \to w \in U(2)$, i.e. it is an $1 \to 2$ correspondence (I will describe it below in details). Hence the "flat" universe is the spinorial $U(2)$. In the case of elementary particles, the real geodetic and shear-free LCR-rays $x^\mu(\sigma)$ pass smoothly from the one $\mathbb{R}^4$-sheet to the other $\mathbb{R}^4$-sheet, creating focussing regions, which will be identified with the particles. Therefore, we have to pay attention to this region, which is going to appear as a "manageable singularity" of the LCR-rays.

We will now show that the physical content of LCR-structure will appear through the emergence of generalized functions. We already know that the smooth LCR-transformations imply the existence of the regular coordinates satisfying the conditions

$$\begin{gathered}
\operatorname{Im} z^0 = \phi_{11}(\overline{z^1}, z^1, \operatorname{Re} z^0)\ ,\ \operatorname{Im} z^{\widetilde{0}} = \phi_{22}(\overline{z^{\widetilde{1}}}, z^{\widetilde{1}}, \operatorname{Re} z^{\widetilde{0}})\ ,\ z^{\widetilde{1}} - \overline{z^1} = \phi_{12}(\overline{z^\beta}, z^{\widetilde{0}}) \\
\phi_{11}(p) = \phi_{22}(p) = \phi_{12}(p) = 0 \quad , \quad d\phi_{11}(p) = d\phi_{22}(p) = d\phi_{12}(p) = 0
\end{gathered} \tag{10.29}$$

in a neighborhood of a point p. But the LCR-structure is a special totally real CR-structure, which at a real analytic neighborhood admits a general analytic transformation $z'^b = f^b(z^c)$, which makes it trivial

$$\operatorname{Im} z'^0 = 0\ ,\ \operatorname{Im} z'^{\widetilde{0}} = 0\ ,\ z'^{\widetilde{1}} - \overline{z'^1} = 0 \tag{10.30}$$

At the non real analytic regions generalized functions appear. We will see that the gravitational, electromagnetic, weak and gluonic dressings of the particle-solitons are distributional representatives with singular supports at these regions.

10.4 The Cartan moving frame approach

In the unbounded realization the LCR-structure is determined by two points of a hypersurface $K(Z^n)$ of $CP(3)$, which satisfy the LCR-conditions (9.1). In homogeneous coordinates it is written as a 4×2 non-degenerate matrix

$$X^{mj} = \begin{pmatrix} \lambda^{Aj} \\ -ir_{A'B}\lambda^{Bj} \end{pmatrix} \tag{10.31}$$

where $r_{A'B}$ is the point of $G(4,2)$ and $\lambda^{Aj}(r_{A'B})$ with $\det(\lambda^{Aj}) \neq 0$ are the projective coordinates of two solutions of $K(Z^n) = 0$. Recall that at the precise point $r_{A'B}$ (line of $CP(3)$) the number of solutions is equal to the degree of the polynomial. Apparently the application of a general 4×4 from the left provides LCR-structures. But the application from the right of a general 2×2 does not give LCR-structure. Hence $X^{mj}(r_{A'B})$ is a section of an $SL(4, \mathbb{C})$ bundle over $G(4,2)$.

The Poincaré×dilation transformation

$$\begin{array}{l} \begin{pmatrix} \lambda' \\ -ir'\lambda' \end{pmatrix} = \begin{pmatrix} B & 0 \\ -iTB & (B^{\dagger})^{-1} \end{pmatrix} \begin{pmatrix} \lambda \\ -ir\lambda \end{pmatrix} \\ \lambda' = B\lambda \quad , \quad r' = (B^{-1})^{\dagger} r B^{-1} + T \\ \det B \neq 0 \quad , \quad T^{\dagger} = T \end{array} \tag{10.32}$$

respects the LCR character of X^{mj}. Its isotropic subroup (for $r_{A'B} = 0$) has $T = 0$. The quotient space is based on the decomposition

$$P = \begin{pmatrix} B & 0 \\ -iTB & (B^{\dagger})^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -iT & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & (B^{\dagger})^{-1} \end{pmatrix} \tag{10.33}$$

The 1-forms of the affine group P are

$$P^{-1}dP = \begin{pmatrix} B^{-1}dB & 0 \\ -iB^{-1}drB & -(B^{-1}dB)^{\dagger} \end{pmatrix} = \begin{pmatrix} e & 0 \\ -iB^{-1}drB & -e^{\dagger} \end{pmatrix} \tag{10.34}$$

and their Cartan structure relations are directly derived.

The present structure $SL(2, \mathbb{C})$ group is not isomorphic to the corresponding Lorentz group of special relativity. It is isomorphic to its proper orthochronus subgroup. Hence we expect to find an explanation of the "time's arrow" observed in nature.

10.5 The coordinate charts of the "flat" LCR-submanifold of $G(4,2)$

We have already made clear that the LCR-structure solution (10.8) is not a proper submanifold of the grassmannian space, because it does not respect the equivalence relation $X \sim X\lambda$ with $\det \lambda \neq 0$. The LCR-structure is rather a section of the tautological vector bundle of $G(4,2)$ viewed as the set of lines X^{ni} in $CP(3)$, where X^{ni}, $i = 1,2$ are two points of a line. But the flat LCR-manifold $X^{\dagger}EX = 0$ is well defined in $G(4,2)$. Besides, perturbative gravity may also be understood as a deformation $r^a = x^a + iy^a(x)$ of the Shilov boundary of the classical domain. Therefore, it is essential to understand the Shilov boundaries of the $SU(1,1)$ and $SU(2,2)$ classical domains.

The usually called complex plane with its infinity $\mathbb{C}\cup\{\infty\}$ is the well defined projective space CP^1 covered with the following two charts

$$\begin{array}{l} Y=\begin{pmatrix} Y^0 \\ Y^1 \end{pmatrix} \quad , \quad Y^0\neq 0 \; or \; Y^1\neq 0 \\ U_1=\{Y^0\neq 0 \; , \; z_1=Y^1(Y^0)^{-1}\in\mathbb{C}\} \\ U_2=\{Y^1\neq 0 \; , \; z_2=Y^0(Y^1)^{-1}\in\mathbb{C}\} \end{array} \tag{10.35}$$

with transition function $z_2=\frac{1}{z_1}$, $z_1\in\mathbb{C}-\{0\}$. The $SU(1,1)$ bounded classical domain is

$$\begin{array}{l} Y^\dagger E_B Y=0 \; , \quad E_B=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \\ D_1=\{1-z_1^\dagger z_1>0 \quad , \; z_1\in U_1\} \\ D_2=\{z_2^\dagger z_2-1>0 \quad , \; z_2\in U_2\} \end{array} \tag{10.36}$$

Notice that the entire classical domain (and its boundary) belongs to one coordinate chart D_1. Therefore it is called the **bounded** realization of the $SU(1,1)$ classical domain.

In the coordinate charts

$$\begin{array}{l} X:=\begin{pmatrix} X^0 \\ X^1 \end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} Y^0 \\ Y^1 \end{pmatrix} \\ with \quad X^0\neq 0 \; or \; X^1\neq 0 \\ U_1'=\{X^0\neq 0 \; , \; r=iX^1(X^0)^{-1}\in\mathbb{C}\} \\ U_2'=\{X^1\neq 0 \; , \; \widehat{r}=iX^0(X^1)^{-1}\in\mathbb{C}\} \end{array} \tag{10.37}$$

the transition function is $\widehat{r}=\frac{-1}{r}$, $r\in\mathbb{C}-\{0\}$. The Cayley transformation is

$$r=i\tfrac{1-z}{1+z} \quad \leftrightarrow \quad z=\tfrac{i-r}{i+r} \tag{10.38}$$

The $SU(1,1)$ classical domain takes the **unbounded** form

$$\begin{array}{l} X^\dagger E_U X=0 \; , \quad E_U=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \\ P_1=\{\frac{r-\bar{r}}{2i}=y>0 \quad , \; r\in U_1\} \\ P_2=\{\frac{\widehat{r}-\bar{\widehat{r}}}{2i}=\widehat{y}<0 \quad , \; \widehat{r}\in U_2\} \end{array} \tag{10.39}$$

In the first chart, it is the unbounded upper half-plane and in the second chart it is the unbounded lower half-plane. Therefore this realization of the classical domain is called **unbounded**. The Cayley transformation of the boundary of the classical domain is

$$\begin{array}{l} r:=x+i0 \quad , \quad z=e^{i\varphi} \\ \mathbb{R}\cup\{\infty\}\ni x=-\cot\frac{\varphi}{2} \quad , \quad \varphi\in(0,2\pi) \end{array} \tag{10.40}$$

where the circle of the disk transforms to the real line plus infinity. The transformation of circles to lines and vice versa is a possibility of the linear fractional transformation $z'=\frac{az+b}{cz+d}$, $ad-bc\neq 0$.

The characteristic boundary of the $SU(2,2)$ classical domain in its **bounded** realization is

$$Y^\dagger E_B Y = Y_1^\dagger Y_1 - Y_2^\dagger Y_2 = 0 \tag{10.41}$$

which, in the "finite" chart $\det Y_1 \neq 0$, has the form $w^\dagger w = 1$. In the "infinity" chart $\det Y_2 \neq 0$ has projective coordinates $\widehat{w} = Y^1(Y^2)^{-1}$ and transition function $\widehat{w} = w^{-1}$. Notice that the boundary of the bounded classical domain $U(2) = U(1) \times SU(2)$ belongs entirely in both charts. Hence in the case of the bounded realization we may work in a proper chart without paying attention to chart transition problems.

The **unbounded** realization of the characteristic boundary of the $SU(2,2)$ classical domain is

$$X^\dagger E_U X = X_2^\dagger X_1 + X_1^\dagger X_2 = 0 \tag{10.42}$$

In the coordinate chart $\det X_1 \neq 0$, with $r \equiv iX_2X_1^{-1}$, it has the form $\frac{r-r^\dagger}{2i} = y = 0$. In the corresponding "infinity" coordinate $\det X_2 \neq 0$, with transition function $\widehat{r} \equiv iX_1X_2^{-1} = -r^{-1}$, it has the form $\frac{\widehat{r}-\widehat{r}^\dagger}{2i} = \widehat{y} = 0$. Notice that in this unbounded realization the transition function of the two boundaries is

$$\begin{array}{l}
\widehat{x} := \begin{pmatrix} \widehat{x}^0 - \widehat{x}^3 & -(\widehat{x}^1 - i\widehat{x}^2) \\ -(\widehat{x}^1 + i\widehat{x}^2) & \widehat{x}^0 + \widehat{x}^3 \end{pmatrix} = -x^{-1} = \frac{-1}{\det x} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \\
\qquad\qquad\qquad\qquad\qquad\qquad\Downarrow \\
\widehat{x}^0 = \frac{-x^0}{\eta_{\rho\sigma}x^\rho x^\sigma} \quad , \quad \widehat{x}^j = \frac{x^j}{\eta_{\rho\sigma}x^\rho x^\sigma} \ , \ j = 1,2,3
\end{array} \tag{10.43}$$

Apparently the boundary of the classical domain $y = 0$ does not belong in the transition region $\det x \neq 0$. Therefore we have to be careful with the projective charts.

Notice that there is an essential difference between these two similar (bounded and unbounded) realizations of the $SU(1,1)$ and $SU(2,2)$ classical domains. The double covering of $SU(2)$ over its homomorphic group $SO(3,\mathbb{R})$ group implies that the group manifold $SU(2)$ needs two non-intersecting sheets $\mathbb{R}^3$ to be covered.

In the $U(2)$ manifold parametrization

$$\begin{array}{l}
w = e^{i\tau} \begin{pmatrix} \cos\rho + i\sin\rho\cos\sigma & -i\sin\rho\sin\sigma\ e^{-i\chi} \\ -i\sin\rho\sin\sigma\ e^{i\chi} & \cos\rho - i\sin\rho\cos\sigma \end{pmatrix} \\
\tau \in (-\pi,\pi) \quad , \quad \rho \in [0,2\pi) \quad , \quad \sigma \in [0,\pi) \quad , \quad \chi \in (0,2\pi)
\end{array} \tag{10.44}$$

the angle variable $\rho \in [0,2\pi)$ takes values in a double region instead of the corresponding angle $\rho \in [0,\pi)$ of the homomorphic $SO(3,\mathbb{R})$ group. Therefore the Cayley transformation needs the following two cartesian coordinates x_+^μ for $\rho \in [0,\pi)$ and x_-^μ for $\rho \in [\pi,2\pi)$ to cover the entire $U(2)$ "flat universe" defined as

$$
\begin{array}{l}
x^0_+ = \frac{\sin\tau}{\cos\tau\ +\ \cos\rho} \\
x^1_+ + ix^2_+ = \frac{\sin\rho}{\cos\tau\ +\cos\rho}\sin\sigma\ e^{i\chi} \\
x^3_+ = \frac{\sin\rho}{\cos\tau\ +\ \cos\rho}\cos\sigma \\
\tau\in(0,2\pi)\ ,\ \rho\in[0,\pi)\ ,\ \sigma\in[0,\pi)\ ,\ \chi\in(0,2\pi) \\
\\
s := \frac{\sin\rho}{\cos\tau\ +\cos\rho} > 0 \quad \leftrightarrow \quad \cos\tau\ +\ \cos\rho > 0
\end{array}
\tag{10.45}
$$

and for $\rho = \pi + \rho'$ as

$$
\begin{array}{l}
x^0_- = \frac{\sin\tau'}{\cos\tau'\ -\ \cos\rho'} \\
x^1_- + ix^2_- = -\frac{\sin\rho'}{\cos\tau'\ -\cos\rho'}\sin\sigma'\ e^{i\chi'} \\
x^3_- = -\frac{\sin\rho'}{\cos\tau'\ -\cos\rho'}\cos\sigma' \\
\tau'\in(-\pi,\pi)\ ,\ \rho'\in[0,\pi)\ ,\ \sigma'\in[0,\pi)\ ,\ \chi'\in(0,2\pi) \\
\\
s' := \frac{-\sin\rho'}{\cos\tau'\ -\cos\rho'} > 0 \quad \leftrightarrow \quad \cos\tau'\ -\ \cos\rho' < 0
\end{array}
\tag{10.46}
$$

It is clear that x^μ_+ and x^μ_- coordinate charts do not overlap, because they correspond to two different regions of $U(2)$, which is the entire "flat universe". In fact the parametrizations are chosen such that x^μ_+ and x^μ_- are coordinate charts of $w = I$ and $-I$ respectively.

One may better undestand that the two $\mathbb{R}^4$ are complementary by noticing that the boundary Cayley transformation (10.14)

$$
w = (iR_0I - x)(iR_0I + x)^{-1} = (iR_0I + x)^{-1}(iR_0I - x) \tag{10.47}
$$

as a transformation from the algebra $x \in su(2)$ to the group $w \in SU(2)$. The proof is very simple

$$
\begin{array}{c}
w^\dagger w = I \\
\Downarrow \\
x^\dagger - x = 0
\end{array}
\tag{10.48}
$$

which is the hermitian condition for the algebra elements of a unitary group.

10.6 Difference of Penrose twistor and LCR-manifold

The reader might have observed the similarities of the LCR-structure formalism with the Penrose twistors. But one should not be confused with the algebraic similarities, because these two formalisms are conceptually completely different! The LCR-structure is a conventional Cartan-Klein structure based on the tetrad-Weyl symmetry. The Penrose twistor program is based on his observation that $CP(3)$ is the space of the solutions of the spinorial partial differential equation

$$
\begin{array}{l}
\nabla^{(A}_{A'}\lambda^{B)} = 0 \\
\\
\lambda^A = \overset{\circ}{\lambda}{}^A - i\widehat{x}^{AA'}\overset{\circ}{w}_{A'}
\end{array}
\tag{10.49}
$$

with

$$Z^m = \begin{pmatrix} \lambda^A \\ w_{B'} \end{pmatrix} \quad , \quad \overline{Z^m} Z^n E_{mn}^{(U)} = 0$$
$$w_{B'} = \overset{\circ}{w}_{B'} \tag{10.50}$$

The hermitian matrix $\widehat{x}^{AA'}$ determines the real line congruence

$$\widehat{x}_{A'A} = \frac{iw_{A'}\overline{w}_A}{\lambda^{C1}\overline{w^1}_C} + r\overline{\lambda^1}_{A'}\lambda^1_A \quad , \quad \forall\ r \quad and \quad \lambda^C \overline{w}_C \neq 0 \tag{10.51}$$

Hence in the context of LCR-manifolds, the Penrose observation may be used in the case of asymptotically flat LCR-manifolds at null infinities, i.e.

$$\begin{array}{l}
\rho_{11}(\overline{X^{m1}}, X^{n1}) = \overline{X^{m1}} X^{n1} E^U_{mn} = 0 \quad , \quad \rho_{22}(\overline{X^{m2}}, X^{n2}) = \overline{X^{m2}} X^{n2} E^U_{mn} = 0 \\
\rho_{12}(\overline{X^{m1}}, X^{n2}) = \overline{X^{m1}} X^{n2} E_{mn} - G_{12}(\overline{X^{m1}}, X^{n2}) = 0 \\
\\
K(X^{m1}) = 0 = K(X^{m2})
\end{array} \tag{10.52}$$

In this case we have two characteristic real vectors (line congruences) $\widehat{x}^a_{(+)}$, $\widehat{x}^a_{(-)}$, with possible different initial points

$$\begin{array}{l}
x_{(+)A'A} = \frac{iw^1_{A'}\overline{w^1}_A}{\lambda^{C1}\overline{w^1}_C} + r\overline{\lambda^1}_{A'}\lambda^1_A \quad , \quad \forall\ r \quad and \quad \lambda^{C1}\overline{w^1}_C \neq 0 \\
\\
x_{(-)A'A} = \frac{iw^2_{A'}\overline{w^2}_A}{\lambda^{C2}\overline{w^2}_C} + s\overline{\lambda^2}_{A'}\lambda^2_A \quad , \quad \forall\ s \quad and \quad \lambda^{C2}\overline{w^2}_C \neq 0
\end{array} \tag{10.53}$$

with fixed z^α and $z^{\widetilde{\alpha}}$ respectively being different. This difference may describe LCR-structure emerging interactions, but apparently the twistor formalism cannot provide the gluonic gauge field.

11 QUADRATIC HYPERSURFACES OF CP^3

We saw that the LCR-structure is directly related to a hypersurface of $CP(3)$. The need for two geodetic and shear-free null congruences implies that the corresponding polynomial must be at least of second degree. After a $SL(4,\mathbb{C})$ transformation a quadratic polynomial takes the form

$$K(X) = \sum_{n=0}^{4} a_n (X^n)^2 \tag{11.1}$$

This surface is regular if

$$\frac{\partial K(X)}{\partial X^m} = 2\begin{pmatrix} a_0X^0 & a_1X^1 & a_2X^2 & a_3X^3 \end{pmatrix} \neq \overrightarrow{0} \tag{11.2}$$

It occurs if $a_n \neq 0$, $\forall n$. Recall that the unique solution $X^n = 0$ is not an element of $CP(3)$, because it does not define a radial line of $\mathbb{C}^4$. If one of a_n vanishes the surface is rank-3 singular and if two of a_n vanishes, it is rank-2 singular. Hence the problem of "flat" LCR-structures is analogous to the number of conics with $RP(3)$ replaced by $CP(3)$.

We will restrict our study to the following set of quadratic hypersurfaces

$$
\begin{array}{l}
K(Z) = A_{mn} Z^m Z^n = 0 \\
A_{mn} \equiv \begin{pmatrix} \omega & P \\ P^\top & 0 \end{pmatrix} \quad , \quad P \equiv \begin{pmatrix} -(p^1 - ip^2) & -p^0 + p^3 \\ p^0 + p^3 & (p^1 + ip^2) \end{pmatrix}
\end{array}
\tag{11.3}
$$

which is invariant under the Poincaré×dilation transformations. The variable p^μ is the (real) 4-momentum and ω a symmetric 2×2 complex matrix. If $\det P = p^\mu p^\nu \eta_{\mu\nu} \neq 0$, the polynomial is irreducible and the hypersurface is regular. After a Poincaré×dilation transformation, it takes the form

$$
K(X^n) = X^1 X^2 - X^0 X^3 + 2a X^0 X^1 = 0 \tag{11.4}
$$

In order to generally study the regularity of a precise point X_0^n, we use the line defined by the point X_0^n and a direction T^n as follows

$$
X^n = X_0^{n1} + sT^n \tag{11.5}
$$

Then the quadratic (in s) polynomial becomes

$$
K(X^n) = K(X_0^n) + 2s \sum_{nm} A_{nm} X_0^n T^m + s^2 \sum_{nm} A_{nm} T^n T^m = 0 \tag{11.6}
$$

If the point is generic, there are two solutions $s_j(T^n),\ j = 1,2$, for any direction T^n. These two solutions represent the two sheets, which correspond to the vertex point X_0^n. The directions with $s_1(T^n) = s_2(T^n)$ appear as singularities of the precise projection. In order to cover this set too, we have to change the vertex X_0^n of the projection.

If the point X_0^n belongs to the quadric we find the quadratic polynomial

$$
K(X^n) = s(2 \sum_{nm} A_{nm} X_0^n T^m + s \sum_{nm} A_{nm} T^n T^m) = 0 \tag{11.7}
$$

As expected, one solution is $s_1 = 0$, and the second one s_2 is different of zero if $\sum\limits_{nm} A_{nm} X_0^n T^m \neq 0$, otherwise the line determined by the solution T_{sol}^n is tangent to the quadric at the point X_0^n. In the present case of regular ($\det(A_{nm}) \neq 0$) quadrics, at each point of the quadric there are two different solutions, which determine the tangent space of the quadric at X_0^n.

Let us now use two points $X^{ni},\ i = 1,2$ of $CP(3)$ to determine the line

$$
X^n = (1-s) X^{n1} + s X^{n2} \tag{11.8}
$$

which intersects the above quadratic hypesurface at two points. Essentially we take $T^n = X^{m2} - X^{n1}$ in the above method. They are the roots of the following quadratic polynomial

$$
K(X^n) = (1-s)^2 K(X^{n1}) + s^2 K(X^{n2}) + 2(1-s)s \sum_{nm} A_{nm} X^{n1} X^{m2} = 0 \tag{11.9}
$$

If the two points belong to the hypersurface we find

$$\begin{aligned}
&K(X^n) = 2(1-s)s \sum_{nm} A_{nm} X^{n1} X^{m2} = 0 \\
&\sum_{nm} A_{nm} X^{n1} X^{m2} \neq 0
\end{aligned} \tag{11.10}$$

that the intersections are only two, $s=0$ and $s=1$, as expected by the Bezout theorem. But if the two points coincide, the method cannot be used, because they cannot define a line of $CP(3)$ or equivalently a plane of $\mathbb{C}^4$.

Recall that every line of $CP(3)$ corresponds to a point $r_{A'B}$ of the grassmannian space $G(4,2)$. If the two columns of the homogeneous coordinates are projectively identified with the two intersection points of the surface with the line $r_{A'B}$, the corresponding sheets $s=0$ and $s=1$ are

$$X_j^n = \begin{pmatrix} 1 \\ \lambda_j(r_{A'B}) \\ -i(r_{0'0} + r_{0'1}\lambda_j) \\ -i(r_{1'0} + r_{1'1}\lambda_j) \end{pmatrix} \quad , \quad j = 1, 2 \tag{11.11}$$

where $\lambda_j(r_{A'B})$ are the two roots of the corresponding Kerr quadratic polynomial. In the case of the simple form (), this polynomial is

$$r_{0'1}\lambda^2 + (r_{0'0} - r_{1'1} + 2ia)\lambda - r_{1'0} = 0 \tag{11.12}$$

and the two roots with their branch curve are

$$\begin{aligned}
&\lambda_{1(2)} = \tfrac{-(r_{0'0} - r_{1'1} + 2ia) \mp \sqrt{(r_{0'0} - r_{1'1} + 2ia)^2 + 4r_{0'1}r_{1'0}}}{2r_{0'1}} \\
&(r_{0'0} - r_{1'1} + 2ia)^2 + 4r_{0'1}r_{1'0} = 0
\end{aligned} \tag{11.13}$$

A variation of the two sheets (branches) of the quadric and the $CP(3)$ embedding of the branch curve can be subsequently found.

11.1 Reducible Poincaré covariant quadrics

If $\det(A_{mn}) = 0$, the quadric is reducible. In the case of Poincaré invariant quadrics we actually study, it happens if $\det P = p^\mu p^\nu \eta_{\mu\nu} = 0$. In this subsection I consider the rank-3 quadric surface. The rank-2 quadric will be studied in connection with the neutrino particle.

After a Lorentz transformation, $p^1 + ip^2 = 0$ may be imposed in order to simplify the calculations. In this case $E - p^3 = 0$ or $E + p^3 = 0$ and the singular points of the algebraic surface are

$$\begin{aligned}
&\begin{pmatrix} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{12} & \omega_{22} & 2E & 0 \\ 0 & 2E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = 0 \quad , \quad E\omega_{11} \neq 0 \\
&Z^0 = Z^1 = Z^2 = 0
\end{aligned} \tag{11.14}$$

for $E - p^3 = 0$. At this point the tangent space is

$$\begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ 0 \end{pmatrix}$$
$$\omega_{11}(X^0)^2 + 2\omega_{12}X^0X^1 + \omega_{22}(X^1)^2 + 4EX^1X^2 = 0 \tag{11.15}$$

It is a tangent cone. Two linearly independent tangent vectors are

$$\begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4E \\ -\omega_{22} \\ 0 \end{pmatrix} \tag{11.16}$$

which determine the tangent plane at the singular point.

The rational parametrization of the tangent cone is found using the pencil of lines that pass from $(0, 1, -\frac{\omega_{22}}{4E})$

$$\begin{array}{l} x_0 = \frac{X^0}{X^1} \quad , \quad x_2 = \frac{X^2}{X^1} \\ \omega_{11}(x_0)^2 + 2\omega_{12}x_0 + \omega_{22} + 4Ex_2 = 0 \\ x_0 = s(x_2 - \frac{\omega_{22}}{4E}) \\ \quad \Downarrow \\ x_0 = \frac{\omega_{22}}{4E} - \frac{4E+2\omega_{12}s}{\omega_{11}s^2} \\ x_2 = -\frac{4E+2\omega_{12}s}{\omega_{11}s} \end{array} \tag{11.17}$$

Hence we see that the reducible quadric is parametrized as follows

$$\begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = c \begin{pmatrix} tx_0 \\ t \\ tx_2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4E(4E + 2\omega_{12}s)st \\ 4E\omega_{11}s^2t \\ [\omega_{11}\omega_{22}s^2 - 4E(4E + 2\omega_{12}s)]t \\ 4E\omega_{11}s^2 \end{pmatrix} \tag{11.18}$$

In the zero gravity approximation we may parametrize the two columns of $G(4, 2)$ as

$$\omega_{11}(Z^0)^2 + 2\omega_{12}Z^0Z^1 + \omega_{22}(Z^1)^2 + 4EZ^1Z^2 = 0$$
$$\begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda \\ -i(x_{0'0} + x_{0'1}\lambda) \\ -i(x_{1'0} + x_{1'1}\lambda) \end{pmatrix} \tag{11.19}$$
$$(\omega_{22} - 4iEx_{0'1})\lambda^2 + 2(\omega_{12} - 2iEx_{0'0})\lambda + \omega_{11} = 0$$

The two roots

$$\lambda_{1(2)} = \frac{-(\omega_{12}-2iEx_{0'0}) \pm \sqrt{(\omega_{12}-2iEx_{0'0})^2 - \omega_{11}(\omega_{22}-4iEx_{0'1})}}{\omega_{22}-4iEx_{0'1}} \tag{11.20}$$

determine the two points on the two branches of the quadric intersected by the line $x_{A'A}$.

For $E + p^3 = 0$ we have the singular points of the algebraic surface

$$\begin{aligned} &\omega_{11}(Z^0)^2 + 2\omega_{12}Z^0Z^1 + \omega_{22}(Z^1)^2 + 4EZ^1Z^2 = 0 \\ &\begin{pmatrix} \omega_{11} & \omega_{12} & 0 & 2E \\ \omega_{12} & \omega_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2E & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = 0 \qquad , \qquad E\omega_{22} \neq 0 \\ &Z^0 = Z^1 = Z^3 = 0 \end{aligned} \tag{11.21}$$

and proceed with $Z^2$and Z^3 interchanged.

Because of the singularity, this reducible surface cannot be properly embedded into $CP(3)$. It has to be blown up at the singularity point, i.e. embedded into $\widehat{M} \times CP^2$. The blowing up at the singularity $(0, 0, 0, 1)$ goes as follows. We first consider

$$\begin{aligned} &\widehat{\Delta} = \{(x_0, x_1, x_2; W^0 : W^1 : W^2) : \\ &\quad : x_0W^1 = x_1W^0, x_0W^2 = x_2W^0, x_1W^2 = x_2W^1\} \subset A^3 \times CP(3) \\ &w_0 = \tfrac{W^0}{W^2} \qquad , \qquad w_1 = \tfrac{W^1}{W^2} \\ &\widehat{\Delta} = \{(x_0, x_1, x_2; w^0, w^1) : x_0 = x_2w^0, x_1 = x_2w^1\} \subset A^3 \times CP^2 \\ &\begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ 0 \end{pmatrix} \\ &\omega_{11}(X^0)^2 + 2\omega_{12}X^0X^1 + \omega_{22}(X^1)^2 + 4EX^1X^2 = 0 \end{aligned} \tag{11.22}$$

Replacing into the surface I find

$$\omega_{11}(x_0)^2 + 2\omega_{12}x_0x_1 + \omega_{22}(x_1)^2 + 4Ex_1x_2 = 0$$

$$\begin{aligned} &\omega_{11}(x_0)^2 + 2\omega_{12}x_0x_1 + \omega_{22}(x_1)^2 + 4Ex_1x_2 = 0 \\ &\widehat{\Delta} = \{(x_0, x_1, x_2; W^0 : W^1 : W^2) : x_0W^1 = x_1W^0, x_0W^2 = x_2W^0, x_1W^2 = x_2W^1\} \subset A^3 \times CP^2 \\ &w_0 = \tfrac{W^0}{W^2} \qquad , \qquad w_1 = \tfrac{W^1}{W^2} \\ &\widehat{\Delta} = \{(x_0, x_1, x_2; w_0, w_1) : x_0 = x_2w_0, x_1 = x_2w_1\} \subset A^3 \times CP^2 \end{aligned} \tag{11.23}$$

We will now study for $E - p^3 = 0$ the rank-2 singular quadric (with the additional condition $\omega_{11} = 0$)

$$Z^1[2\omega_{12}Z^0 + \omega_{22}Z^1 + 4EZ^2] = 0 \qquad , \qquad E\omega_{12} \neq 0 \tag{11.24}$$

Its singular points are

$$\begin{pmatrix} 0 & \omega_{12} & 0 & 0 \\ \omega_{12} & \omega_{22} & 2E & 0 \\ 0 & 2E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = 0 \tag{11.25}$$

$$Z^1 = 0 = \omega_{12} Z^0 + 2EZ^2$$

This hypersurface of $CP(3)$ is the union of two independent hyperplanes. Their singularities are the points of their intersection line.

For $E + p^3 = 0$ and $\omega_{22} = 0$, the singular points of the algebraic surface are

$$\begin{gathered} Z^0[\omega_{11} Z^0 + 2\omega_{12} Z^1 + 4EZ^3] = 0 \quad , \quad E\omega_{12} \neq 0 \\ \begin{pmatrix} \omega_{11} & \omega_{12} & 0 & 2E \\ \omega_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2E & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = 0 \\ Z^0 = Z^1 = \omega_{12} Z^1 + 2EZ^3 \end{gathered} \tag{11.26}$$

and proceed with $Z^2$and Z^3 interchanged.

12 AUTOMORPHISMS OF LCR-STRUCTURES

The automorphisms of a structure are diffeomorphic deformations which preserve the precise form of the structure. For example, looking for "static axially symmetric spacetimes" is equivalent to look for metrics, which admit the "automorphisms of time-translation and z-rotation". The general mathematical approach to the mathematical goes through the Lie derivative along the deformation vector and apply the implied conditions on the structure variables. In the present case we may apply either the Cartan method ([4]) or the direct method of the real tetrad subject to the tetrad-Weyl symmetry.

The Cartan method starts with the extended tetrad (in the ambient complex manifold) written in structure coordinates

$$\ell = \ell^{\widetilde{\alpha}} \partial_{\widetilde{\alpha}} \quad , \quad m = m^{\widetilde{\alpha}} \partial_{\widetilde{\alpha}} \quad , \quad n = n^{\alpha} \partial_{\alpha} \quad , \quad \overline{m} = \overline{m}^{\alpha} \partial_{\alpha} \tag{12.1}$$

The generator G of an automorphism of this LCR-structure must satisfy the conditions

$$\begin{gathered} [G, \ell] = f_{\widetilde{0}}^{\widetilde{0}} \ell + f_{\widetilde{0}}^{\widetilde{1}} m \quad , \quad [G, m] = f_{\widetilde{1}}^{\widetilde{0}} \ell + f_{\widetilde{1}}^{\widetilde{1}} m \\ [G, n] = f_0^0 n + f_0^1 \overline{m} \quad , \quad [G, \overline{m}] = f_1^0 n + f_1^1 \overline{m} \end{gathered} \tag{12.2}$$

This implies that the coordinates of a (real) symmetry generator have the following dependence on the structure coordinates

$$G = G^{\alpha}(z^{\beta}) \partial_{\alpha} + G^{\widetilde{\alpha}}(z^{\widetilde{\beta}}) \partial_{\widetilde{\alpha}} + \overline{G^{\alpha}(z^{\beta})} \overline{\partial_{\alpha}} + \overline{G^{\widetilde{\alpha}}(z^{\widetilde{\beta}})} \overline{\partial_{\widetilde{\alpha}}} \tag{12.3}$$

In the direct method, we look for diffeomorphic transformations which leave the LCR-tetrad invariant. If $X^{\mu}\partial_{\mu}$ is the general form of the symmetry algebra generator of these transformations, their Lie derivatives must satisfy the relations

$$L_X \ell = \Lambda_X \ell \quad , \quad L_X n = N_X n \quad , \quad L_X m = M_X m \tag{12.4}$$

where Λ_X, N_X, M_X are the tetrad-Weyl parameters. If we apply the property

$$L_X(f\omega) = f L_X \omega + (X^{\mu}\partial_{\mu} f)\omega \tag{12.5}$$

of the Lie derivative relative to the general form of the symmetry algebra generator, the symmetry conditions () are reduced to

$$L_X \ell = 0 \quad , \quad L_X n = 0 \quad , \quad L_X m = 0 \tag{12.6}$$

which imply

$$L_{X_j} \ell = 0 \quad , \quad L_{X_j} n = 0 \quad , \quad L_{X_j} m = 0 \tag{12.7}$$

for all the independent generators of the Lie algebra.

Let us now suppose that $X_j^{\mu}\partial_{\mu}$ are the independent Lie generators, which determine the symmetry algebra of the LCR-structure

$$[X_i^{\mu}\partial_{\mu} \ , \ X_j^{\mu}\partial_{\mu}] = h_{ijl} X_l^{\mu}\partial_{\mu} \tag{12.8}$$

Using the formula

$$\begin{array}{l}
L_X(e^a) = X \lrcorner (de^a) + d(X \lrcorner e^a) \\
L_X(f\omega \wedge \varpi) = L_X(f)\omega \wedge \varpi + f L_X(\omega) \wedge \varpi + f\omega \wedge L_X \varpi \\
\\
dL_{X_j} \ell = L_{X_j}(d\ell) = 0 \quad , \quad dL_{X_j} n = L_{X_j}(dn) = 0 \quad , \quad dL_{X_j} m = dL_{X_j}(dm) = 0
\end{array} \tag{12.9}$$

and the integrability conditions which are defined by the relations

$$\begin{array}{l}
d\ell = -(\varepsilon + \overline{\varepsilon})\ell \wedge n + (\alpha + \overline{\beta} - \overline{\tau})\ell \wedge m + (\overline{\alpha} + \beta - \tau)\ell \wedge \overline{m} + (\rho - \overline{\rho})m \wedge \overline{m} \\
dn = -(\gamma + \overline{\gamma})\ell \wedge n + (\pi - \alpha - \overline{\beta})n \wedge m + (\overline{\pi} - \overline{\alpha} - \beta)n \wedge \overline{m} + (\mu - \overline{\mu})m \wedge \overline{m} \\
dm = -(\tau + \overline{\pi})\ell \wedge n + (\gamma - \overline{\gamma} + \overline{\mu})\ell \wedge m + (\varepsilon - \overline{\varepsilon} - \rho)n \wedge m + (\beta - \overline{\alpha})m \wedge \overline{m}
\end{array} \tag{12.10}$$

We finally find the relations

$$\begin{array}{l}
L_{X_j}(\varepsilon + \overline{\varepsilon}) = 0 \quad , \quad L_{X_j}(\alpha + \overline{\beta} - \overline{\tau}) = 0 \quad , \quad L_{X_j}(\rho - \overline{\rho}) = 0 \\
L_{X_j}(\gamma + \overline{\gamma}) = 0 \quad , \quad L_{X_j}(\pi - \alpha - \overline{\beta}) = 0 \quad , \quad L_{X_j}(\mu - \overline{\mu}) = 0 \\
L_{X_j}(\tau + \overline{\pi}) = 0 \quad , \quad L_{X_j}(\gamma - \overline{\gamma} + \overline{\mu}) = 0 \quad , \quad L_{X_j}(\varepsilon - \overline{\varepsilon} - \rho) = 0 \quad , \quad L_{X_j}(\beta - \overline{\alpha}) = 0
\end{array} \tag{12.11}$$

We know that the elementary particles belong to representations of the Poincaré group. In the context of PCFT the elementary particles (leptons and quarks) are LCR-structure (and gauge field) distributional configurations, which transform under a Poincaré transformation. The configurations, which admit the time-translation and z-rotation generators as their automorphisms, will be considered as their "eigenstates", and the configurations of the representation will be derived by simply applying the general transformation. Hence the problem of finding the elementary particles turns out to find the LCR-structures, which are automorphic relative to the infinitesimal t-translation and z-rotation.

Using the following structure coordinates, viewed as regular structure coordinates satisfying the following embedding conditions

$$z^{0}:=i\tfrac{X^{21}}{X^{01}}\quad,\quad z^{1}:=\tfrac{X^{11}}{X^{01}}\quad,\quad z^{\widetilde{0}}:=i\tfrac{X^{32}}{X^{12}}\quad,\quad z^{\widetilde{1}}:=-\tfrac{X^{02}}{X^{12}}$$

$$\tfrac{z^{0}-\overline{z^{0}}}{2i}-U(\tfrac{z^{0}+\overline{z^{0}}}{2},z^{1},\overline{z^{1}})=0\quad,\quad z^{\widetilde{1}}-Z(z^{\widetilde{0}},\overline{z^{0}},\overline{z^{1}})=0\quad,\quad \tfrac{z^{\widetilde{0}}-\overline{z^{\widetilde{0}}}}{2i}-V(\tfrac{z^{\widetilde{0}}-\overline{z^{\widetilde{0}}}}{2},z^{\widetilde{1}},\overline{z^{\widetilde{1}}})=0 \tag{12.12}$$

the infinitesimal z-rotations are

$$\begin{array}{l}\delta X^{0i}=-i\frac{\varepsilon^{12}}{2}X^{0i}\quad,\quad \delta X^{1i}=i\frac{\varepsilon^{12}}{2}X^{1i}\quad,\quad \delta X^{2i}=-i\frac{\varepsilon^{12}}{2}X^{2i}\quad,\quad \delta X^{3i}=i\frac{\varepsilon^{12}}{2}X^{3i}\\ \delta z^{0}=0\quad,\quad \delta z^{1}=i\varepsilon^{12}z^{1}\quad,\quad \delta z^{\widetilde{0}}=0\quad,\quad \delta z^{\widetilde{1}}=-i\varepsilon^{12}z^{\widetilde{1}}\end{array} \tag{12.13}$$

the above general form is restricted to

$$\tfrac{z^{0}-\overline{z^{0}}}{2i}-U(\tfrac{z^{0}+\overline{z^{0}}}{2},z^{1}\overline{z^{1}})=0\quad,\quad z^{\widetilde{1}}-\overline{z^{1}}W(z^{\widetilde{0}},\overline{z^{0}})=0\quad,\quad \tfrac{z^{\widetilde{0}}-\overline{z^{\widetilde{0}}}}{2i}-V(\tfrac{z^{\widetilde{0}}-\overline{z^{\widetilde{0}}}}{2},z^{\widetilde{1}}\overline{z^{\widetilde{1}}})=0 \tag{12.14}$$

Assuming a quadratic Kerr polynomial $K(X^m)=\sum\limits_{m,n}A_{mn}X^mX^n$, I find that the automorphic one relative to the z-rotation has the form

$$K=A_{01}X^{0}X^{1}+A_{03}X^{0}X^{3}+A_{12}X^{1}X^{2}+A_{23}X^{2}X^{3} \tag{12.15}$$

The infinitesimal time-translations are

$$\begin{array}{l}\delta X^{0i}=0\quad,\quad \delta X^{1i}=0\quad,\quad \delta X^{2i}=-i\varepsilon^{0}X^{0i}\quad,\quad \delta X^{3i}=-i\epsilon^{0}X^{1i}\\ \delta z^{0}=\varepsilon^{0}\quad,\quad \delta z^{1}=0\quad,\quad \delta z^{\widetilde{0}}=\varepsilon^{0}\quad,\quad \delta z^{\widetilde{1}}=0\end{array} \tag{12.16}$$

The automorphic LCR-structure conditions relative to both z-rotation and time-translation become

$$\tfrac{z^{0}-\overline{z^{0}}}{2i}-U(z^{1}\overline{z^{1}})=0\quad,\quad z^{\widetilde{1}}-\overline{z^{1}}W(z^{\widetilde{0}}-\overline{z^{0}})=0\quad,\quad \tfrac{z^{\widetilde{0}}-\overline{z^{\widetilde{0}}}}{2i}-V(z^{\widetilde{1}}\overline{z^{\widetilde{1}}})=0 \tag{12.17}$$

and the quadratic Kerr polynomial $K(X^m)$ takes the form

$$K=A_{01}X^{0}X^{1}+A_{12}(X^{1}X^{2}-X^{0}X^{3}) \tag{12.18}$$

Notice that if we try to impose the dilation

$$\begin{array}{ll}\delta X^{0i}=-\frac{\varepsilon}{2}X^{0i}\quad, & \delta X^{1i}=-\frac{\varepsilon}{2}X^{1i}\\ \delta X^{2i}=\frac{\varepsilon}{2}X^{2i}\quad, & \delta X^{3i}=\frac{\varepsilon}{2}X^{3i}\end{array} \tag{12.19}$$

as an additional automorphism, the definition (12.12) of the structure coordinates implies the infinitesimal transformation

$$\begin{array}{ll}\delta z^{0}=\varepsilon z^{0}\quad, & \delta z^{1}=0\\ \delta z^{\widetilde{0}}=\varepsilon z^{\widetilde{0}}\quad, & \delta z^{\widetilde{1}}=0\end{array} \tag{12.20}$$

which imposes $a = 0$, which is the "spherical" degenerate LCR-structure. The additional condition of asymptotic flatness at null infinity

$$X^{m1}E_{mn}X^{n1} = 0 = X^{m2}E_{mn}X^{n2}$$

$$\frac{z^0 - \overline{z^0}}{2i} + 2a\frac{z^1\overline{z^1}}{1+z^1\overline{z^1}} = 0 \quad , \quad z^{\widetilde{1}} - \overline{z^1}W(z^{\widetilde{0}} - \overline{z^0}) = 0 \quad , \quad \frac{z^{\widetilde{0}} - \overline{z^{\widetilde{0}}}}{2i} - 2a\frac{z^{\widetilde{1}}\overline{z^{\widetilde{1}}}}{1+z^{\widetilde{1}}\overline{z^{\widetilde{1}}}} = 0 \tag{12.21}$$

and a symmetry between the left and right chiral columns $z^1\overline{z^1} = z^{\widetilde{1}}\overline{z^{\widetilde{1}}}$. Its embedding in $G(4,2)$ is

$$\begin{array}{l}
X^{mi} = \begin{pmatrix} 1 & -z^{\widetilde{1}} \\ z^1 & 1 \\ -iz^0 & iz^{\widetilde{1}}(z^{\widetilde{0}} - 2ia) \\ -iz^1(z^0 + 2ia) & -iz^{\widetilde{0}} \end{pmatrix} \\
z^0 = t - f_0(r) - 2ia\sin^2\frac{\theta}{2} \quad , \quad z^1 = e^{i\varphi}e^{-iaf_1(r)}\tan\frac{\theta}{2} \\
z^{\widetilde{0}} = t + f_0(r) + 2ia\sin^2\frac{\theta}{2} \quad , \quad z^{\widetilde{1}} = e^{-i\varphi}e^{-iaf_1(r)}\tan\frac{\theta}{2} \\
f_0(r) = \int\frac{r^2+a^2}{\Delta}dr \quad , \quad f_1(r) = \int\frac{1}{\Delta}dr
\end{array} \tag{12.22}$$

The important point here is that this LCR-structure with "electromagnetic" and "gravitational" dressing admits as automorphisms the two commuting generators of the Poincaré group.

12.1 Symmetric LCR-structures

In the unbounded realization the affine group of the boundary of the $SU(2,2)$ classical domain is the Poincaré×dilation group

$$\begin{array}{l}
\begin{pmatrix} X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} B & 0 \\ -iTB & (B^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\
\det B \neq 0 \quad , \quad T^\dagger = T
\end{array} \tag{12.23}$$

The infinitesimal z-rotations are

$$\delta X^0 = -i\frac{\varepsilon^{12}}{2}X^0 \quad , \quad \delta X^1 = i\frac{\varepsilon^{12}}{2}X^1 \quad , \quad \delta X^2 = -i\frac{\varepsilon^{12}}{2}X^2 \quad , \quad \delta X^3 = i\frac{\varepsilon^{12}}{2}X^3 \tag{12.24}$$

The regular symmetric quartic $\sum A_{klmn}Z^kZ^lZ^mZ^n$ homogeneous Kerr polynomial is

$$\begin{array}{l}
A_{0011}(Z^0)^2(Z^1)^2 + A_{0013}(Z^0)^2(Z^1)(Z^3) + A_{0033}(Z^0)^2(Z^3)^2 + \\
\quad + A_{0112}(Z^0)(Z^1)^2(Z^2) + A_{0123}(Z^0)(Z^1)(Z^2)(Z^3) + \\
\quad + A_{0233}(Z^0)(Z^2)(Z^3)^2 + A_{1122}(Z^1)^2(Z^2)^2 + \\
\quad + A_{1223}(Z^1)(Z^2)^2(Z^3) + A_{2233}(Z^2)^2(Z^3)^2 = 0
\end{array} \tag{12.25}$$

If we impose automorphism under an infinitesimal massive time-translation

$$\delta X^0 = 0 \quad , \quad \delta X^1 = 0 \quad , \quad \delta X^2 = -i\varepsilon^0 X^0 \quad , \quad \delta X^3 = -i\epsilon^0 X^1 \tag{12.26}$$

to the above axially symmetric Kerr polynomial, we find that the invariant polynomial takes the form

$$A(Z^1Z^2 - Z^0Z^3)^2 + B(Z^0Z^1)(Z^1Z^2 - Z^0Z^3) + C(Z^0Z^1)^2 = 0 \tag{12.27}$$

which is essentially a reducible quartic polynomial equivalent to the product of two static quadrics.

The general quartic has four intersections with a general line of $CP(3)$. Two such intersections determine an LCR-structure

$$X^{ni} = \begin{pmatrix} \lambda^{Ai} \\ -ir_{A'B}\lambda^{Ai} \end{pmatrix} \tag{12.28}$$

at a complex point $r_{A'B} = x_{A'B} + iy_{A'B}$ of the grassmannian manifold $G(4,2)$. Two different $SL(2,\mathbb{C})$ transformations can apply on λ^{Ai}

$$\lambda'^{Ai} = S'^{A}_{B}\lambda^{Bi} \quad , \quad \lambda''^{Aj} = \lambda^{Bi}S''^{j}_{i} \tag{12.29}$$

The first is simply part of a Lorentz transformation implying the form of the solution in the new coordinates. But the second transformation gives a linear combination of the two solutions at the same grassmannian point, which are no longer solutions of the Kerr polynomial. In the spinorial terminology of general relativity, the second transformation changes the dyad basis. The new one is no longer geodetic and shear-free, and hence they do not define a LCR-structure.

In the zero gravity case

$$X^n = \begin{pmatrix} 1 \\ \lambda \\ -i[(t-z) - (x-iy)\lambda] \\ -i[-(x+iy) + (t+z)] \end{pmatrix} \tag{12.30}$$

in the $X^0 \neq 0$ coordinate chart, the axisymmetric quartic Kerr polynomial takes the form

$$\begin{aligned} &A_{0011}(Z^0)^2(Z^1)^2 + A_{0013}(Z^0)^2(Z^1)(Z^3) + A_{0033}(Z^0)^2(Z^3)^2 + \\ &+A_{0112}(Z^0)(Z^1)^2(Z^2) + A_{0123}(Z^0)(Z^1)(Z^2)(Z^3) + \\ &+A_{0233}(Z^0)(Z^2)(Z^3)^2 + A_{1122}(Z^1)^2(Z^2)^2 + \\ &+A_{1223}(Z^1)(Z^2)^2(Z^3) + A_{2233}(Z^2)^2(Z^3)^2 = 0 \end{aligned} \tag{12.31}$$

13 RULED SURFACES OF $CP(3)$

Through the realizability of the lorentzian CR-structure (the fundamental geometric structure of PCFT) and its projectivization, we found its direct relation with the (reducible and irreducible) algebraic surfaces of $CP(3)$. Our first result was the identification of the physical Poincaré group with the linear transformation of the boundary of the Siegel domain, viewed as a subgroup of $SU(2,2)$, which is subgroup of $SL(4,\mathbb{C})$ of $CP(3)$. It is known that besides the implicit (polynomial) parameterization, a surface has the explicit (not necessarily rational) parameterization. The ruled surfaces have a special explicit parameterization, which reveals their internal property to be made up of lines, which determine a generally complex trajectory in the grassmannian space. That is, they have the form

$$\begin{aligned}
&Z^m(\tau, s) = (1-s)Z^{m1}(\tau) + sZ^{m2}(\tau) = \\
&\quad = Z^{m1}(\tau) + sT^m(\tau) \\
&T^m(\tau) := Z^{m2}(\tau) - Z^{m1}(\tau)
\end{aligned} \tag{13.1}$$

where $T^m(\tau)$ indicates the direction of the generating line which meets $Z^{m1}(\tau)$ (the generatrix) at τ. In this definition the Kerr function is replaced with its proper parametrization.

Any two different directrices of a ruled surface are in one-to-one correspondence implied by their intersection points with a generating line (generator, ruling). Hence all the curves of the ruled surface have the same genus, which is the genus of the ruled surface.

Let us now consider a ruled surface of order k. A non-tangent plane in general position in $CP(3)$, passing through a generator l, also intersects the surface with a curve $Z^{ml}(\tau)$ of order $k-1$. The line l intersects $D^m(\tau)$ in $k-1$ points, which are the points where the plane meets other generators of the ruled surface. Hence every generator is met by $k-2$ other generators (see Edge's book[8]). Besides, there is a double curve $D^m(\tau)$, where every two generators meet and it meets every generator in $k-2$ points.

The generating lines correspond to complex points of the grassmannian manifold $G(4,2)$, with projective coordinates

$$\begin{aligned}
&\xi(\tau) =: iX_2X_1^{-1} =: \begin{pmatrix} \xi^0 - \xi^3 & -(\xi^1 - i\xi^2) \\ -(\xi^1 + i\xi^2) & \xi^0 + \xi^3 \end{pmatrix} \\
&X_1 =: \begin{pmatrix} Z^{01} & Z^{02} \\ Z^{11} & Z^{12} \end{pmatrix} = \begin{pmatrix} Z^0(\tau,0) & Z^0(\tau,1) \\ Z^0(\tau,0) & Z^1(\tau,1) \end{pmatrix} \\
&X_2 =: \begin{pmatrix} Z^{21} & Z^{22} \\ Z^{31} & Z^{32} \end{pmatrix} = \begin{pmatrix} Z^2(\tau,0) & Z^2(\tau,1) \\ Z^3(\tau,0) & Z^3(\tau,1) \end{pmatrix}
\end{aligned} \tag{13.2}$$

Using homogeneous coordinates, this curve of $G(4,2)$ is spanned by the two points $Z^{ni}(\tau);\ i = 1,2$ of $\mathbb{C}^4$. The curve is called non-degenerate if the following determinant does not identically vanish

$$\begin{aligned}
&\det[Z^{n1}, Z^{n2}, \tfrac{dZ^{n1}}{d\tau}, \tfrac{dZ^{n2}}{d\tau}] = \det\begin{pmatrix} X_1 & \dot{X}_1 \\ -i\xi X_1 & -i(\dot{\xi}X_1 + \xi\dot{X}_1) \end{pmatrix} = \\
&= \det[\begin{pmatrix} 1 & 0 \\ -i\xi & 1 \end{pmatrix}\begin{pmatrix} X_1 & \dot{X}_1 \\ 0 & -i\dot{\xi}X_1 \end{pmatrix}] = -\det(\dot{\xi})(\det X_1)^2
\end{aligned} \tag{13.3}$$

This happens if and only if $\dot{\xi}^a \dot{\xi}^b \eta_{ab} \neq 0$. This condition will differentiate the massive from the massless partner (neutrino) of a leptonic generation. The complex trajectory is related to the ordinary classical trajectory of the particle viewed as a soliton. If they are real, they are identified with the well known trajectories of the Lienard-Wiechert potential. In the context of general relativity, Newman first observed the relation of such complex trajectories with the geodetic and shear-free null congruences.

If the curve is degenerate, the gaussian curvature of the ruled surface vanishes and the ruled surface is called developable. The developable surfaces of $CP(3)$ are cones, cylinders and tangent developables with $T^n(\tau) = \frac{dZ_1^m(\tau)}{d\tau}$.

In the context of general relativity, Newman[?] showed that a complex trajectory in complex Minkowski spectime defines a geodetic and shear free null congruence. A quite general Poincaré covariant explicit parameterization of a ruled hypersurface of $CP(3)$ and its corresponding grassmannian patch, is

$$X^{ni} = \begin{pmatrix} Z^0(\tau,s) & Z^0(\widetilde{\tau},\widetilde{s}) \\ Z^1(\tau,s) & Z^1(\widetilde{\tau},\widetilde{s}) \\ Z^2(\tau,s) & Z^2(\widetilde{\tau},\widetilde{s}) \\ Z^3(\tau,s) & Z^3(\widetilde{\tau},\widetilde{s}) \end{pmatrix} = \begin{pmatrix} \lambda^{Ai} \\ -ir_{B'B}\lambda^{Bi} \end{pmatrix} \tag{13.4}$$

where the $r_{B'B} = r_b \sigma^b_{B'B}$ are the projective coordinates, generally outside the $\xi_b(\tau)$ trajectory of the ruled surface. Because simply not all the pairs of points of a ruled surface belong to rulings. If $r_b \in \xi_b(\tau)$ a projective line of $CP(3)$ cincides with a ruling line of the ruled surface.

The reparametrization (τ) ambiguity may be fixed with either the condition $\xi^0(\tau) = \tau$ or the more restrictive one $\dot{\xi}^a \dot{\xi}^b \eta_{ab} = 1$. In the coordinate chart $Z^0 = 1$ of $CP(3)$, a general point of the ruled surface determined by a trajectory $\xi^b(\tau)$ has the form

$$Z^n(\tau,\lambda) = \begin{pmatrix} 1 \\ 0 \\ -i(\xi^0 - \xi^3) \\ i(\xi^1 + i\xi^2) \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ i(\xi^1 - i\xi^2) \\ -i(\xi^0 + \xi^3) \end{pmatrix} \tag{13.5}$$

$$\lambda =: \frac{(1-s)\lambda^{11}(\tau) + s\lambda^{12}(\tau)}{(1-s)\lambda^{01}(\tau) + s\lambda^{02}(\tau)}$$

The first term is the "directrix curve" of the ruled surface and the second is the "generating line" ("ruling") of the surface. This is the form we have already assumed in order to introduce the trajectory of the electron. And precisely the linear trajectory $\xi^b(\tau) = v^b\tau + d^b$ with $(v^b)^2 = 1$ corresponds to the "free" electron and with $(v^b)^2 = 0$ corresponds to its neutrino.

A general line ($r \in G(4,2)$) of $CP(3)$, generally intersects d times the ruled surface and d coincides with the algebraic degree of the ruled surface. Two of these intersection points $(X^{n1}(\tau_1, s_1)\ ,\ X^{n2}(\tau_2, s_2))$ determine the line and subsequently

$$X^{ni} = \begin{pmatrix} \lambda^{A1}(\tau_1, s_1) & \lambda^{A2}(\tau_2, s_2) \\ -i\xi_{B'B}(\tau_1)\lambda^{B1} & -i\xi_{B'B}(\tau_2)\lambda^{B2} \end{pmatrix} = \begin{pmatrix} \lambda^{Ai} \\ -ir_{B'B}\lambda^{Bi} \end{pmatrix}$$
$$\lambda^{Ai}(\tau_i, s_i) = \begin{pmatrix} \lambda^{0i}(\tau_i, s_i) \\ \lambda^{1i}(\tau_i, s_i) \end{pmatrix} \quad , \quad i = 1, 2 \tag{13.6}$$

Using spinorial coordinates, the above relation takes the form

$$\begin{pmatrix} \lambda^{Aj} \\ -ir_{A'B}\lambda^{Bj} \end{pmatrix} = \begin{pmatrix} \lambda^{Aj} \\ -i\xi_{A'B}(\tau_j)\lambda^{Bj} \end{pmatrix}$$
$$(r_{A'B} - \xi_{A'B}(\tau_j))\lambda^{Bj} = 0 \tag{13.7}$$

which are two homogeneous linear equations for every $j = 1, 2$. They admit a (projectively) non-vanishing solution λ^{Bj} for every j, if

$$\begin{aligned} \det(r_{A'B} - \xi_{A'B}(\tau)) = \det(r - \xi(\tau)) = \\ = (r^a - \xi^a)(r^b - \xi^b)\eta_{ab} = 0 \end{aligned} \tag{13.8}$$

Every generally complex solution $\tau(r)$ of this equation is replaced back into (13.7) and find the corresponding spinor λ^A. For every column of X^{ni} (point of $CP(3)$) we get a pair of generally complex functions

$$\begin{aligned} z^0(r) = \tau_1(r) \quad &, \quad z^1(r) = \tfrac{\lambda^{11}(\tau_1(r))}{\lambda^{01}(\tau_1(r))} \\ \widetilde{z^0}(r) = \tau_2(r) \quad &, \quad \widetilde{z^1}(r) = -\tfrac{\lambda^{01}(\tau_2(r))}{\lambda^{11}(\tau_2(r))} \end{aligned} \tag{13.9}$$

which may be assumed as the structure coordinates in the ambient complex manifold of the LCR-structure. After the projection to the real LCR-submanifold, they become proper structure coordinates.

The reader must not confuse the set of "straight" lines of $CP(3)$, which are all the points $r_{A'A}$ of the grassmannian manifold $G(4,2)$, with the rulings of the rulled surface, i.e. the "straight" lines which belong (as sets of points) to the ruled hypersurface of $CP(3)$, and which are just the points of the complex trajectory $\xi_{A'A}(\tau)$ in $G(4,2)$.

A geometric visualization of the above mathematical procedure is the following. A point of the grassmannian manifold with projective coordinates r determines a line of $CP(3)$. This line intersects the hypersurface of $CP(3)$ at a number of points (equal to the polynomial degree of the surface), which belong to different sheets of the surface of $CP(3)$. Every pair of intersection points with homogeneous coordinates X^{ni} may be taken as the corresponding homogeneous coordinates of the grassmannian point r. Hence every point $r \in G(4,2)$, determines (and is determined by) two points $\xi^b(\tau_1)$ and $\xi^b(\tau_2)$ of the complex trajectory $\xi^b(\tau)$ with two corresponding spinors $\lambda^{A1}(\tau_1, s_1)$ and $\lambda^{A2}(\tau_2, s_2)$. Notice that the trajectory $\xi^b(\tau)$ determines the algebraic subcurve of $CP(3)$, where the two sheets intersect, $\lambda^A(\tau_1) = \lambda^A(\tau_2)$. This is valid for any point r of the ambient complex manifold. If the point belongs in the real LCR-submanifold, i.e. if $r^a = x^a + iy^a(x)$, the "observer" at the point x^a, has the local null system

$$L^a = \tfrac{1}{\sqrt{2}}\sigma^a_{A'A}\overline{\lambda}^{A'1}\lambda^{A1}, \quad N^a = \tfrac{1}{\sqrt{2}}\sigma^a_{A'A}\overline{\lambda}^{A'2}\lambda^{A2}, \quad M^a = \tfrac{1}{\sqrt{2}}\sigma^a_{A'A}\overline{\lambda}^{A'2}\lambda^{A1} \tag{13.10}$$

where we have substituted $r^a = x^a + iy^a(x)$. Notice that if we choose the $\xi^0(\tau) = \tau$ parameterization, we find

$$\begin{array}{ll} z^0(x) = r^0(x) - \sqrt{(r^i(x) - \xi^i(z^0))^2} \quad , & z^1(x) = \frac{r^1 + ir^2 - \xi^1(z^0) - i\xi^2(z^0)}{r^0 + r^3 - \xi^0(z^0) - \xi^3(z^0)} \\ z^{\widetilde{0}}(x) = r^0(x) + \sqrt{(r^i(x) - \xi^i(z^{\widetilde{0}}))^2} \quad , & z^{\widetilde{1}}(x) = \frac{r^1 - ir^2 - \xi^1(z^{\widetilde{0}}) + i\xi^2(z^{\widetilde{0}})}{r^0 - r^3 - \xi^0(z^{\widetilde{0}}) + \xi^3(z^{\widetilde{0}})} \end{array} \tag{13.11}$$

Hence, the left column of X^{n1} provides the retarded coordinates $z^\alpha(x)$ and the right column X^{n2} provides the advanced coordinates $z^{\widetilde{\alpha}}(x)$. The (curved) LCR-tetrad is found as usual by simply taking the differential forms of the structure coordinates and using their reality conditions.

One can easily see that in the zero gravity approximation $y^a(x) = 0$, the structure coordinates (and the null tetrad) are completely determined by the generally complex trajectory as we should expect from Kerr's theorem (in Minkowski spacetime). In the first $\frac{1}{c}$ approximation, the LCR-structure coordinates take the form

$$\begin{array}{ll} z^0(x) \simeq x^0 - \frac{1}{c}\sqrt{(x^i - \xi^i(x^0))^2} \quad , & z^1(x) \simeq \frac{x^1 + ix^2 - \xi^1(x^0) - i\xi^2(x^0)}{x^0 + x^3 - \xi^0(x^0) - \xi^3(x^0)} \\ z^{\widetilde{0}}(x) \simeq x^0 + \frac{1}{c}\sqrt{(x^i - \xi^i(x^0))^2} \quad , & z^{\widetilde{1}}(x) \simeq \frac{x^1 - ix^2 - \xi^1(x^0) + i\xi^2(x^0)}{x^0 - x^3 - \xi^0(x^0) + \xi^3(x^0)} \end{array} \tag{13.12}$$

where the (dimensional) light velocity factor is made apparent in order to reveal the newtonian approximation.

The points of the trajectory $\xi^i(x^0)$ are the singularities of the structure coordinates. If the trajectory is real, the trajectory is just a curve in LCR-manifold. But if the trajectory is complex $\xi^j(x^0) = \xi^j_R(x^0) + i\xi^j_I(x^0)$, the singularity is the surface

$$\begin{array}{l} (x^i - \xi^i(x^0))^2 = (x^j - \xi^j_R(x^0))^2 - (\xi^j_I(x^0))^2 - 2i(x^j - \xi^j_R(x^0))\xi^j_I(x^0) = 0 \\ \\ (x^j - \xi^j_R(x^0))^2 - (\xi^j_I(x^0))^2 = 0 \quad , \quad (x^j - \xi^j_R(x^0))\xi^j_I(x^0) = 0 \end{array} \tag{13.13}$$

This is the well-known ring-like singularity of Kerr-type metrics in general relativity. The imaginary part of the trajectory is related to the spin of the LCR-structure. It is exactly this imaginary part that generates the fermionic gyromagnetic ratio of the Kerr-Newman spacetime. Hence the complex trajectory $\xi^a(\tau)$ is the singular curve where two sheets of the surface of $CP(3)$ intersect.

The free electron and its neutrino correspond to the linear trajectory $\xi^b(\tau) = v^b\tau + d^b$ with v^b real and d^b generally complex. Using the common parametrization $\xi^0 = s$, i.e. $\xi^j(s) = V^j s + d^b$ we may realize how the electron and its neutrino singularities are related. The electron is a ruled irreducible surface with non-vanishing gaussian curvature with its algebraic singularity the cylinder space ring $\forall x^0 \in \mathbb{R}$. Its neutrino is a reducible developable hypersurface, which is the union of two hyperplanes, intersecting at a line which closes at the "infinity" of the compact reducible hypersurface of $CP(3)$.

Up to now we have considered that both left and right columns of the homogeneous coordinates have the same generally complex trajectory. But we may also assume that they have different trajectories. That is

$$\begin{pmatrix} \lambda^{Aj} \\ -ir_{A'B}\lambda^{Bj} \end{pmatrix} = \begin{pmatrix} \lambda^{Aj} \\ -i\xi^{(j)}_{A'B}(\tau_j)\lambda^{Bj} \end{pmatrix} \tag{13.14}$$

Recall that in the implicit (polynomial) parameterization of the hypersurfaces of $CP(3)$, we had considered the possibility of irreducible and reducible polynomials. Apparently the case of different trajectories for each column corresponds to the case of reducible polynomial surfaces.

13.1 The free electron trajectory

Let us now consider the simple case of LCR-structures with linear trajectory

$$\xi^a(\tau) = v^a\tau + c^a \quad , \quad (\dot{\xi}^a)^2 = (v^a)^2 = 1$$

$$\begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} \lambda^0 \\ \lambda^1 \\ -i[((v^0 - v^3)\tau + (c^0 - c^3))\lambda^0 - ((v^1 - iv^2)\tau + (c^1 - ic^2))\lambda^1 \\ -i[-((v^1 + iv^2)\tau + (c^1 + ic^2))\lambda^0 + ((v^0 + v^3)\tau + (c^0 + c^3))\lambda^1 \end{pmatrix} \tag{13.15}$$

where v^a is the real velocity of the LCR-structure and c^a is generally complex. If we eliminate the projective variable λ^A, we find

$$\begin{pmatrix} iX^2 - (c^0 - c^3)X^0 + (c^1 - ic^2)X^1 \\ iX^3 + (c^1 + ic^2)X^0 - (c^0 + c^3)X^1 \end{pmatrix} = \begin{pmatrix} (v^0 - v^3)X^0 - (v^1 - iv^2)X^1 \\ -(v^1 + iv^2)X^0 + (v^0 + v^3)X^1 \end{pmatrix}\tau \tag{13.16}$$

and after the elimination of the second variable τ, we find the following quadratic hypersurface

$$\begin{array}{l}
A_{mn}X^m X^n = 0 \\
\\
A_{00} = -2i[(c^0 - c^3)(v^1 + iv^2) - (c^1 + ic^2)(v^0 - v^3)] \\
A_{01} = 2i[c^0v^3 - c^3v^0 + ic^1v^2 - ic^2v^1] \\
A_{02} = -(v^1 + iv^2) \quad , \quad A_{03} = -(v^0 - v^3) \\
A_{11} = -2i[(c^1 - ic^2)(v^0 + v^3) - (c^0 - c^3)(v^1 - iv^2)] \\
A_{12} = v^0 + v^3 \quad , \quad A_{13} = v^1 - iv^2 \quad , \quad A_{22} = A_{23} = A_{33} = 0
\end{array} \tag{13.17}$$

Recall that the Poincaré transformation of the homogeneous coordinates X, the velocity v and complex initial position are given by the formula

$$\begin{aligned} &\begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} B & 0 \\ -iTB & (B^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ &v' = (B^{-1})^\dagger v(B^{-1}) \quad , \quad c' = (B^{-1})^\dagger c(B^{-1}) + T \\ &\det B = 1 \quad , \quad T^\dagger = T \end{aligned} \tag{13.18}$$

After a spacetime translation and a Lorentz transformation we may impose $c^b = (0, 0, 0, ia)$ and $v = I$. Then the quadratic polynomial takes the form

$$X^1 X^2 - X^0 X^3 + 2aX^0 X^1 = 0 \tag{13.19}$$

which is the Kerr polynomial already suggested by the use of automorphisms. Notice that a complex Poincaré transformation, which preserves the LCR-structure form, but it does not preserve the classical domain, removes completely c^a.

13.2 Complex trajectories in Plucker coordinates

I consider the six homogeneous Plucker coordinates of $G(4, 2)$,

$$\begin{aligned} &p_{mn} = \epsilon_{mnkl} X^{k1} X^{l2} \\ &\epsilon^{mnkl} p_{mn} p_{kl} = 0 \end{aligned} \tag{13.20}$$

which is a quadric in CP^5. In the proper homogeneous coordinates it takes the simple form

$$\begin{aligned} &R^0 = i\tfrac{p_{23}-p_{14}}{2} \quad , \quad R^1 = i\tfrac{p_{24}-p_{13}}{2} \quad , \quad R^2 = -\tfrac{p_{24}+p_{13}}{2} \\ &R^3 = i\tfrac{p_{23}+p_{14}}{2} \quad , \quad R^4 = \tfrac{p_{12}-p_{34}}{2} \quad , \quad R^5 = \tfrac{p_{12}+p_{34}}{2} \\ &\epsilon^{mnkl} p_{mn} p_{kl} = (R^0)^2 - (R^1)^2 - (R^3)^2 + (R^5 - R^4)(R^5 + R^4) = 0 \end{aligned} \tag{13.21}$$

Using the formula

$$\begin{aligned} &\epsilon^{A'B'} \sigma^\mu_{A'A} \sigma^\nu_{B'B} = \Sigma^{[\mu\nu]}_{(AB)} + \eta^{\mu\nu} \epsilon_{AB} \\ &\lambda^{A1} \lambda^{B2} \epsilon_{AB} = 1 \end{aligned} \tag{13.22}$$

we find

$$\begin{aligned} &p_{12} = -\eta_{ab} r^a r^b \quad , \quad p_{13} = i(r^1 + ir^2) \quad , \quad p_{14} = i(r^0 - r^3) \\ &p_{23} = -i(r^0 + r^3) \quad , \quad p_{24} = -i(r^1 - ir^2) \quad , \quad p_{34} = 1 \\ &R^0 = r^0, \quad R^1 = r^1, \quad R^2 = r^2, \quad R^3 = r^3, \quad R^4 = -\tfrac{\eta_{ab} r^a r^b + 1}{2}, \quad R^5 = -\tfrac{\eta_{ab} r^a r^b - 1}{2} \end{aligned} \tag{13.23}$$

where the notation has been chosen such that R^j becomes real for "flat" LCR structures.

The LCR-surface in Plucker coordinates (in the quadric of CP^5) is found by simply replacing $r^a = x^a + iy^a(x)$. But the LCR-structure cannot be completely determined, because it also depends on the sections $\lambda_1^A$and λ_2^A in CP^1 fiber. In the case of zero gravity the LCR-surface is implied by $y^a(x) \equiv 0$, as usual.

In the case of a ruled surface determined (in my notation) by the two curves

$$X_1^n(\tau, 0) = \begin{pmatrix} 1 \\ 0 \\ -i(\xi^0 - \xi^3) \\ i(\xi^1 + i\xi^2) \end{pmatrix} \quad , \quad X_2^n(\tau, 1) = \begin{pmatrix} 1 \\ 1 \\ -i(\xi^0 - \xi^3) + i(\xi^1 - i\xi^2) \\ i(\xi^1 + i\xi^2) - i(\xi^0 + \xi^3) \end{pmatrix} \tag{13.24}$$

the corresponding trajectory has the form

$$\begin{array}{l} p_{12} = -\eta_{ab}\xi^a\xi^b \quad , \quad p_{13} = i(\xi^1 + i\xi^2) \quad , \quad p_{14} = i(\xi^0 - \xi^3) \\ p_{23} = -i(\xi^0 + \xi^3) \quad , \quad p_{24} = -i(\xi^1 - i\xi^2) \quad , \quad p_{34} = 1 \\ \\ T^0 = \xi^0 \quad , \quad T^1 = \xi^1 \quad , \quad T^2 = \xi^2 \quad , \quad T^3 = \xi^3 \end{array} \tag{13.25}$$

in the Klein quadric of CP^5.

13.3 Classification of rational ruled surfaces in $CP(3)$

The are determined by one of its generatrices $Z^{n1}(\tau)$ and the direction $T^n(\tau)$ of the corresponding to τ line, which belongs to the ruled surface

$$Z^m(\tau, s) = Z^{m1}(\tau) + sT^m(\tau) \tag{13.26}$$

If the direction $T^n(\tau)$ is tangent to a generatrix, i.e. $T^n(\tau) = \frac{dZ^{n1}(\tau)}{d\tau}$, the ruled surface becomes developable and constitutes the formal massive-massless pairs of leptons and quarks.

A typical classification of the ruled surfaces (and its related developable) exists through the classification of the curves in $CP(3)$. The most general rational (genus $g = 0$) curve has the form

$$\begin{array}{l} Z^n(\tau) = \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \tau^3 \end{pmatrix} \quad , \quad Z^n(\tau, s) = \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \tau^3 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ t_1(\tau) \\ t_2(\tau) \end{pmatrix} \\ \\ Z^0 Z^2 - (Z^1)^2 = 0, \quad (Z^0)^2 Z^3 - (Z^1)^3 = 0, \quad Z^1 Z^3 - (Z^2)^2 = 0 \end{array} \tag{13.27}$$

where the last line gives the algebraic form of this cubic curve. The corresponding developable surface is

$$Z^n(\tau, s) = \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \tau^3 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 2\tau \\ 3\tau^2 \end{pmatrix} \tag{13.28}$$

where the direction of lines is the holomorphic derivative of the generatrix.

13.4 Frenet and Darboux frames in $CP(3)$

Following Griffiths[17] the homogeneous coordinates $Z^n \in \mathbb{C}^4$ of the points of $CP(3)$ are understood relative to a unitary basis $\{A_n,\ n = 0, 1, 2, 3\}$. Thus the set of all these frames are the group $SU(4)$. Then (following Cartan) we have

$$\begin{array}{l} dA_n = A_m \theta^m_n \quad , \quad \theta^m_n = \overline{\theta^n_m} \\ \\ \Omega = d\theta^k_n + \theta^k_m \wedge \theta^m_n = 0 \end{array} \tag{13.29}$$

where θ^m_n is the $SU(4)$ connection and Ω is the vanishing of the group curvature (the Maurer-Catan equations). Fixing the vector A_0, the group $SU(4)$ is reduced down to $U(1) \times SU(3)$, where $U(1)$ applies along the line determined by A_0, and $SU(3)$ in its perpendicular $(A_0^{\perp})$ 3-dimensional space.

In terms of a holomorphic curve $Z(\zeta)$ of $CP(3)$, we may build up a Frenet frame[17] if its jacobian

$$J(\zeta) := Z(\zeta) \vee Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta) \vee Z^{(3)}(\zeta) \tag{13.30}$$

does not identically vanish. In the neighborhood of a regular point $J(\zeta_0) \neq 0$, the (complex) orthonormal basis is

$$\begin{array}{l} W_0 = e^{i\chi_0} \frac{Z(\zeta)}{||Z(\zeta)||} \\ W_0 \vee W_1 = e^{i\chi_1} \frac{Z(\zeta) \vee Z^{(1)}(\zeta)}{||Z(\zeta) \vee Z^{(1)}(\zeta)||} \\ W_0 \vee W_1 \vee W_2 = e^{i\chi_2} \frac{Z(\zeta) \vee Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta)}{||Z(\zeta) \vee Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta)||} \\ W_0 \vee W_1 \vee W_2 \vee W_3 = e^{i\chi_3} \frac{Z(\zeta) \vee Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta) \vee Z^{(3)}(\zeta)}{||Z(\zeta) \vee Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta) \vee Z^{(3)}(\zeta)||} \end{array} \tag{13.31}$$

where the angle ambiguity is explicitly written. Thus the Frenet equations are

$$\begin{array}{l} dW_0 = \theta_{00} W_0 + \theta_{01} W_1 \\ dW_1 = \theta_{10} W_0 + \theta_{11} W_1 + \theta_{12} W_2 \\ dW_2 = \theta_{21} W_1 + \theta_{22} W_2 + \theta_{23} W_3 \\ dW_3 = \theta_{32} W_2 + \theta_{33} W_3 \end{array} \tag{13.32}$$

In the affine subspace $Z^0 = 1$, the holomorphic curve is $Z^i(\zeta)$, the group is restricted to $SU(3)$ and the corresponding affine $ASU(3)$ basis is

$$\begin{array}{l} e^{i\chi_1} W_1 = \frac{Z^{(1)}(\zeta)}{||Z^{(1)}(\zeta)||} \\ e^{i\chi_2} W_1 \vee W_2 = \frac{Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta)}{||Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta)||} \\ e^{i\chi_3} W_1 \vee W_2 \vee W_3 = \frac{Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta) \vee Z^{(3)}(\zeta)}{||Z^{(1)}(\zeta) \vee Z^{(2)}(\zeta) \vee Z^{(3)}(\zeta)||} \end{array} \tag{13.33}$$

and the Cartan lift of the holomorphic curve $Z^i(\zeta)$ is

$$\zeta \quad \rightarrow \quad g(\zeta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ Z^1 & W^1_1 & W^1_2 & W^1_3 \\ Z^2 & W^2_1 & W^2_2 & W^2_3 \\ Z^3 & W^3_1 & W^3_2 & W^3_3 \end{pmatrix} \tag{13.34}$$

The flat $ASU(3)$ connection ω and its flat curvature are

$$\begin{array}{l} \omega := g^{-1}dg \\ \Omega = d\omega + \omega \wedge \omega = 0 \end{array} \tag{13.35}$$

That is

$$\begin{array}{l} dg = g\omega \\ dZ = W_i \omega^i \quad , \quad dW_i = W_j \omega_i^j \\ \qquad\qquad\qquad \Downarrow \\ d\omega^i + \omega_k^i \wedge \omega^k = 0 \quad , \quad d\omega_j^i + \omega_k^i \wedge \omega_j^k = 0 \end{array} \tag{13.36}$$

In the case of non-trivial scalar product $< \overline{W_i} \cdot W_j >= \delta_{ij}$, the Maurer-Cartan connection of ω_j^i will satisfy the anti-hermiticity condition

$$\begin{array}{l} < \overline{W_i} \cdot W_j >= \delta_{ij} \\ \qquad\qquad \Downarrow \\ < d\overline{W_i} \cdot W_j > + < \overline{W_i} \cdot dW_j >= 0 \\ \qquad\qquad \Downarrow \\ \delta_{il}\omega_j^l + \delta_{jk}\overline{\omega_i^k} = 0 \end{array} \tag{13.37}$$

That is ω_j^i belongs to the Lie group of $SU(3)$, as expected.

In the simple case of the 1-dimensional conformal metric and the corresponding (1,1) form

$$\begin{array}{l} ds^2 = h^2 d\varsigma d\overline{\zeta} \quad , \quad \Omega = \frac{i}{2} h^2 d\varsigma \wedge d\overline{\zeta} \\ \\ Ric\Omega = i\partial\overline{\partial} \log h \end{array} \tag{13.38}$$

where the last is the Ricci form (curvature). Let a (1,0)-form θ such that

$$\begin{array}{l} \theta = h d\zeta \\ \Omega = \frac{i}{2} \theta \wedge \overline{\theta} \end{array} \tag{13.39}$$

symmetric relative to $\theta \to e^{i\psi}\theta$. Then there is the (antihermitian) connection

$$\begin{array}{l} \phi = -\partial \log h + \overline{\partial} \log h = -\overline{\phi} \\ d\theta = \phi \wedge \theta \\ \\ Ric\Omega = \frac{i}{2} d\phi \end{array} \tag{13.40}$$

providing the $U(1)$ curvature, which is apparently invariant under the $U(1)$ transformation

$$\begin{array}{l} \theta \to e^{i\psi}\theta \\ \phi \to \phi + d\psi \end{array} \tag{13.41}$$

In the case of $CP(3)$ we have[17] the hermitian Frenet frame $W_k(\zeta)$ is a linear combination of the first k derivatives of the holomorphic curve

$$\begin{array}{l} W_0(\zeta),\ W_1(\zeta),\ W_2(\zeta),\ W_3(\zeta) \\ \\ dW_0 = \theta_{00} W_0 + \theta_{01} W_1 \\ dW_1 = \theta_{10} W_0 + \theta_{11} W_1 + \theta_{12} W_2 \\ dW_2 = \theta_{21} W_1 + \theta_{22} W_2 + \theta_{23} W_3 \\ dW_3 = \theta_{32} W_2 + \theta_{33} W_3 \end{array} \tag{13.42}$$

Let

$$\begin{array}{ll} \Omega_0 = \frac{i}{2}\theta_{01} \wedge \theta_{01} & \to \ \phi_0 = \theta_{00} - \theta_{11} \\ \Omega_1 = \frac{i}{2}\theta_{12} \wedge \theta_{12} & \to \ \phi_1 = \theta_{11} - \theta_{22} \\ \Omega_2 = \frac{i}{2}\theta_{23} \wedge \theta_{23} & \to \ \phi_2 = \theta_{22} - \theta_{33} \\ \Omega_3 = \frac{i}{2}\theta_{01} \wedge \theta_{01} & \end{array} \tag{13.43}$$

the (1,1) and the corresponding (antihermitian) connections

The definition of the Darboux frame starts from the choice of a curve in a surface, and it is finally adapted to the surface by completing the frame with the normal vectors of the surface. Our interest to this kind of frames comes from the possibility to "clarify" (at least at the classical level) how a free massless neutrino may appear as massive. The classical emergence of a particle trajectory strongly suggests that the original hypersurface of $CP(3)$, has to be (or related to) a ruled surface which is intimately related to a complex trajectory. The massive and massless electron and neutrino pair will be determined with a proper ruled surface and its corresponding developable (tangential) surface respectively. Therefore we will now determine the Darboux frame of a ruled surface

$$X^m(\tau, s) = X^{m1}(\tau) + sT^m(\tau) \tag{13.44}$$

The Frenet frame is based on the base curve $X^{m1}(\tau)$, which provides the orthonormalization of the four linearly independent X^{m1}, X'^{m1}, X''^{m1}, X'''^{m1}, using the Gram-Schmidt method. In the case of a developable surface, the Frenet frame cannot be defined because the Gaussian curvature vanishes. But its adaptation to the surface works. Notice that the two tangent vectors to the ruled surface are

$$\begin{array}{l} \partial_\tau X^m(\tau, s) = \partial_\tau X^{m1}(\tau) + s\partial_\tau T^m(\tau) \\ \partial_s X^m(\tau, s) = T^m(\tau) \end{array} \tag{13.45}$$

For a tangential developable surface i.e. $T^m(\tau) = \partial_\tau X^{m1}(\tau)$, we have

$$\begin{array}{l} \partial_\tau X^m(\tau, s) = \partial_\tau X^{m1}(\tau) + s\partial_\tau^2 X^{m1}(\tau) \\ \partial_s X^m(\tau, s) = \partial_\tau X^{m1}(\tau) \end{array} \tag{13.46}$$

the surface is well defined for $s \neq 0 \neq \partial_\tau^2 X^{m1}$. The third vector is chosen vertical to the above vectors (i.e. to the surface).

14 DISCRETE TRANSFORMATIONS

We know that a quite general LCR-structure may be viewed as a line in $CP(3)$ which intersects an (reducible or irreducible) algebraic hypersurface in d distinct points. d is the degree of the hypersurface. Two distinct intersection points X^{mi} determine the structure coordinates if

$$\begin{array}{l} \rho_{11}(\overline{X^{m1}}, X^{n1}) = 0 = \rho_{22}(\overline{X^{m2}}, X^{n2}) \\ \rho_{12}(\overline{X^{m1}}, X^{n2}) = 0 \\ K_1(X^{m1}) = 0 = K_2(X^{m2}) \end{array} \tag{14.1}$$

where all the functions are (independently) homogeneous relative to X^{n1} and X^{n2}.

A general symmetry group of the above set of solutions is $SL(4,\mathbb{C})$, the symmetry group of the grassmannian space $G(4,2)$. Do not confuse the symmetries of the set of solutions and the automorphisms of a given LCR-structure. The symmetry group of the boundary of the classical domain (the flat spacetime in PCFT) is

$$\begin{aligned}
&\begin{pmatrix} X_1'^i \\ X_2'^i \end{pmatrix} = \begin{pmatrix} B & 0 \\ -iTB & (B^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} X_1^i \\ X_2^i \end{pmatrix} \\
&B \in SL(2,\mathbb{C}) \quad , \quad T^\dagger = T
\end{aligned} \tag{14.2}$$

In the context of riemannian geometry a general Lorentz transformation is defined as the set of matrices g, which preserve the Minkowski metric ($g^T \eta g = \eta$) and they constitute the general Lorentz group. The complete Lorentz group leaves invariant all the three regions of the Minkowski spacetime divided by the light cone

$$\begin{array}{lllll}
x^a \eta_{ab} x^b > 0 & , & x^0 > 0 & , & forward\ time-like\ cone \\
x^a \eta_{ab} x^b > 0 & , & x^0 < 0 & , & backward\ time-like\ cone \\
x^a \eta_{ab} x^b < 0 & , & space-like\ region & &
\end{array} \tag{14.3}$$

The subgroup with $\det g = 1$, is called proper Lorentz group.

A minkowiskian vector x^a is represented with a hermitian 2×2 matrix $x_{A'B} = \eta_{ab} x^b \sigma^a_{A'B}$ with measure $\det x = x^a x^b \eta_{ab}$ and vice-versa a hermitian matrix defines a vector $x^c = \sigma^{cA'B} x_{A'B}$. Hence the $SL(2,\mathbb{C})$ transformation B is a general Lorentz transformation $x' = B^\dagger x B$, because it preserves hermiticity and $\det x$. In fact B and $-B$ correspond to the same general Lorentz transformation. But $SL(2,\mathbb{C})$ is a connected group, therefore it must correspond to the proper Lorentz group ($\det g = 1$), where the identity of the general Lorentz group belongs. Hence the symmetry of PCFT is just the proper orthochronus Lorentz group, while the symmetry of the Minkowski space is the larger general Lorentz. That is LCR-structure may "see" spatial and temporal reflections, which cannot be "seen" by the metric of the riemannian geometry.

14.1 Parity (Spatial reflection)

The spatial reflection is the cartesian coordinate x^a transformation with the matrix

$$s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{14.4}$$

It does not belong to the proper Lorentz group because $\det g = -1$. In fact the complete Lorentz group is the group obtained by including s into the proper Lorentz group.

The spatial reflection s defines the following automorphism of the proper Lorentz group $g' = sgs^{-1}$. This automorphism is external for proper Lorentz group and internal for the complete and general Lorentz groups. Note that

$$(g^T)^{-1} = \eta g \eta^{-1} = sgs^{-1} \tag{14.5}$$

The spatial reflection s does not belong to $SL(2,\mathbb{C})$, which is identical with the proper Lorentz group. That is there is no element of $SL(2,\mathbb{C})$, which corresponds to s. But one can easily see that

$$x' = \begin{pmatrix} x'^0 - x'^3 & -(x'^1 - ix'^2) \\ -(x'^1 + ix'^2) & x'^0 + x'^3 \end{pmatrix} = \begin{pmatrix} x^0 + x^3 & (x^1 - ix^2) \\ (x^1 + ix^2) & x^0 - x^3 \end{pmatrix} = \epsilon \overline{x} \epsilon^{-1}$$

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{14.6}$$

We know that $g'_{\pm a'} = sg_{\pm a}s^{-1}$ is an external automorphism of the proper Lorentz group, the corresponding automorphism of the $SL(2,\mathbb{C})$ group is $A' = \pm(A^\dagger)^{-1}$, that is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad A' = (A^\dagger)^{-1} = \pm \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} \tag{14.7}$$

The geodetic and shear-free condition for a flat null vector $\ell^\mu = \overline{\lambda}^{A'} \sigma^\mu_{A'B} \lambda^B$ is

$$\begin{gathered} \lambda^A \lambda^B \nabla_{A'A} \lambda_B = 0 \quad , \quad \lambda^A = \lambda^0 \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \\ \Updownarrow \\ (\partial_{0'0}\lambda) + \lambda(\partial_{0'1}\lambda) = 0 \;\; and \;\; (\partial_{1'0}\lambda) + \lambda(\partial_{1'1}\lambda) = 0 \end{gathered} \tag{14.8}$$

A general solution of these equations is any $\lambda(x^{A'B}) = \frac{\lambda^1}{\lambda^0}$, which satisfies a relation

$$K(\lambda,\; x_{0'0} + x_{0'1}\lambda,\; x_{1'0} + x_{1'1}\lambda) = 0 \tag{14.9}$$

with an arbitrary Kerr function $K(\cdot,\ \cdot,\ \cdot)$. Be careful on the upper and lower indices

$$\begin{aligned} x_{A'A} &= x_\mu \sigma^\mu_{A'A} = \begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} \\ x^{A'A} &= x^\mu \sigma^{A'A}_\mu = \begin{pmatrix} x^0 + x^3 & (x^1 + ix^2) \\ (x^1 - ix^2) & x^0 - x^3 \end{pmatrix} \\ \partial_{A'A} &= \sigma^\mu_{A'A} \partial_\mu = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix} = 2\frac{\partial}{\partial x^{A'A}} \end{aligned} \tag{14.10}$$

A parity transformation implies

$$
\begin{aligned}
&\tfrac{\partial\lambda'}{\partial x'^{0'0}}+\lambda'\tfrac{\partial\lambda'}{\partial x'^{0'1}}=0 \qquad and \qquad \tfrac{\partial\lambda'}{\partial x'^{1'0}}+\lambda'\tfrac{\partial\lambda'}{\partial x'^{1'1}}=0\\
&\qquad\qquad\qquad\qquad\Updownarrow\\
&\lambda'=\tfrac{-1}{\overline{\lambda}} \quad , \quad \lambda'^{A}=\lambda'^{0}\begin{pmatrix}1\\ \lambda'\end{pmatrix}=-\tfrac{\lambda'^{0}}{\overline{\lambda}}\begin{pmatrix}-\overline{\lambda}\\ 1\end{pmatrix}\\
&\tfrac{\partial\lambda}{\partial x^{0'0}}+\lambda\tfrac{\partial\lambda}{\partial x^{0'1}}=0 \qquad and \qquad \tfrac{\partial\lambda}{\partial x^{1'0}}+\lambda\tfrac{\partial\lambda}{\partial x^{1'1}}=0
\end{aligned}
\tag{14.11}
$$

Notice that the representation of the spinor changes, as expected, because parity does not belong to $SL(2,\mathbb{C})$.

In the context of LCR-structures, parity transformation takes the form

$$
\begin{aligned}
&X'^{mi}=\begin{pmatrix}\lambda'^{Ai}\\ -ix'_{A'B}\lambda'^{Ai}\end{pmatrix}=\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix}\begin{pmatrix}\widetilde{\lambda}\\ -i\overline{x}\widetilde{\lambda}\end{pmatrix}\epsilon^{-1}\\
&\widetilde{\lambda}=\epsilon^{-1}\lambda'\epsilon
\end{aligned}
\tag{14.12}
$$

which is the interchange of the intersection points of the line of $CP(3)$ of the conjugate spacetime point with the hypersurface $K(Z^{n})=0$.

14.2 Temporal reflection

The temporal reflection matrix is

$$
t=\begin{pmatrix}-1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1\end{pmatrix}
\tag{14.13}
$$

This matrix transforms the forward time-like region to the backward time-like region and vice-versa. It does not belong nor to the proper Lorentz group, neither to the complete Lorentz group. The general Lorentz group is the group obtained by including t into the complete Lorentz group.

The temporal reflection t defines the following automorphism of the proper Lorentz group $g'=tgt^{-1}$.This automorphism is external for the proper ($\det g=1$) and the complete (preserves the light-cone distinguishing connected regions) Lorentz group, but it is internal for the general Lorentz group. Note that

$$
(g^{T})^{-1}=\eta g\eta^{-1}=sgs^{-1}=tgt^{-1}
\tag{14.14}
$$

One can easily see that a temporal reflection implies

$$
\begin{aligned}
&x'=\begin{pmatrix}x'^{0}-x'^{3} & -(x'^{1}-ix'^{2})\\ -(x'^{1}+ix'^{2}) & x'^{0}+x'^{3}\end{pmatrix}=\begin{pmatrix}-x^{0}-x^{3} & -(x^{1}-ix^{2})\\ -(x^{1}+ix^{2}) & -x^{0}+x^{3}\end{pmatrix}=-\epsilon\overline{x}\epsilon^{-1}\\
&\epsilon=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}
\end{aligned}
\tag{14.15}
$$

A temporal reflection transformation implies

$$\begin{aligned}
&\frac{\partial\lambda'}{\partial x'^{0'0}}+\lambda'\frac{\partial\lambda'}{\partial x'^{0'1}}=0 \qquad and \qquad \frac{\partial\lambda'}{\partial x'^{1'0}}+\lambda'\frac{\partial\lambda'}{\partial x'^{1'1}}=0\\
&<==>\\
&\lambda'=\frac{-1}{\overline{\lambda}} \quad , \quad \lambda'^{A}=\lambda'^{0}\begin{pmatrix}1\\ \lambda'\end{pmatrix}=-\frac{\lambda'^{0}}{\overline{\lambda}}\begin{pmatrix}-\overline{\lambda}\\ 1\end{pmatrix}\\
&\frac{\partial\lambda}{\partial x^{0'0}}+\lambda\frac{\partial\lambda}{\partial x^{0'1}}=0 \qquad and \qquad \frac{\partial\lambda}{\partial x^{1'0}}+\lambda\frac{\partial\lambda}{\partial x^{1'1}}=0
\end{aligned} \tag{14.16}$$

Notice that the representation of the spinor changes, as expected, because t does not belong to $SL(2,\mathbb{C})$.

In the context of LCR-structures, temporal reflection transformation takes the form

$$\begin{aligned}
&X'^{mi}=\begin{pmatrix}\lambda'^{Ai}\\ -ix'_{A'B}\lambda'^{Ai}\end{pmatrix}=\begin{pmatrix}-\epsilon & 0\\ 0 & -\epsilon\end{pmatrix}\begin{pmatrix}\widetilde{\lambda}\\ -i\overline{x}\widetilde{\lambda}\end{pmatrix}\epsilon^{-1}\\
&\widetilde{\lambda}=\epsilon^{-1}\lambda'\epsilon
\end{aligned} \tag{14.17}$$

which is the interchange of the intersection points of the line of $CP(3)$ of the conjugate spacetime point with the hypersurface $K(Z^n)=0$.

14.3 Left and right chiral parts

The fundamental quantity of general relativity, metric $g_{\mu\nu}=:\eta_{ab}e^{a}_{\mu}e^{b}_{\nu}$, does not uniquely define the tetrad e^{a}_{μ}. It is defined up to a local Lorentz transformation. On the other hand, LCR-structure starts with a precise geodetic and shear-free tetrad and the ambiguity is transferred on the definition of the metric. Therefore, at this point, substantial differences are expected between PCFT and general relativity, because the local Lorentz transformation does not preserve the LCR-structure property of the tetrad. In order to realize these differences we first consider the case of flat spacetimes. The null tetrad of the Minkowski metric is determined by two independent spinors λ^{Ai} through the relations

$$\begin{aligned}
&\ell^{\mu}=\overline{\lambda}^{A'1}\sigma^{\mu}_{A'B}\lambda^{B1} \quad , \quad m^{\mu}=\overline{\lambda}^{A'1}\sigma^{\mu}_{A'B}\lambda^{B2}\\
&n^{\mu}=\overline{\lambda}^{A'2}\sigma^{\mu}_{A'B}\lambda^{B2} \quad , \quad \overline{m}^{\mu}=\overline{\lambda}^{A'2}\sigma^{\mu}_{A'B}\lambda^{B1}
\end{aligned} \tag{14.18}$$

Any linear transformation $\lambda'^{Ai}=C^{i}_{j}\lambda^{Aj}$ with $\det(C^{i}_{j})=1$ implies a different null tetrad of the Minkowski metric. But this linear transformation does not preserve the non-linear geodetic and shear-free conditions. Kerr theorem states that only transformations between roots of homogeneous polynomials

$$K(\lambda^{A},\ x_{0'0}\lambda^{0}+x_{0'1}\lambda^{1},\ x_{1'0}\lambda^{0}+x_{1'1}\lambda^{1})=0 \tag{14.19}$$

connect geodetic and shear-free null tetrads.

In the context of PCFT the LCR-tetrad $(\ell,m;n,\overline{m})$ is separated into the two parts (ℓ,m) and $(n,\overline{m})$, which are separately integrable but they are related with a complex condition. The first pair (ℓ, m) is called left part and the second pair $(n, \overline{m})$ is called right part. The left-right chiral transformation is the interchange $(\ell, m) \Leftrightarrow (n, \overline{m})$ of these two integrable pairs of the LCR-tetrad. If the LCR-structure is realizable, an equivalent definition is the interchange of the two pairs $(z^{\alpha}) \Leftrightarrow (z^{\widetilde{\beta}})$ of the structure coordinates. If the LCR-manifold is a real surface of the grassmannian space $G(4,2)$, an equivalent definition of left-right chiral transformation is the interchange of the two columns $X^{n1} \Leftrightarrow X^{m2}$ of its homogeneous coordinates, which are two intersection points of the corresponding line and an hypersurface of $CP(3)$.

These two parts are related with the corresponding spinor parts through their $SL(2,\mathbb{C})$ transformations. It appears through the regular coordinates of a LCR-structure which satisfy the conditions

$$\begin{array}{l} \operatorname{Im} z^{0} = \phi_{11}(\overline{z^{1}}, z^{1}, \operatorname{Re} z^{0}) \quad , \quad \operatorname{Im} z^{\widetilde{0}} = \phi_{22}(\overline{z^{\widetilde{1}}}, z^{\widetilde{1}}, \operatorname{Re} z^{\widetilde{0}}) \\ z^{\widetilde{1}} - \overline{z^{1}} = \phi_{12}(\overline{z^{\beta}}, z^{\widetilde{0}}) \\ \\ \phi_{11}(0) = \phi_{22}(0) = \phi_{12}(0) = 0 \quad , \quad d\phi_{11}(0) = d\phi_{22}(0) = d\phi_{12}(0) = 0 \end{array} \tag{14.20}$$

Notice that we may assume the variables $\operatorname{Re} z^{0}, z^{1}, \operatorname{Re} z^{\widetilde{0}}$ as independent coordinates, where z^{1} is the complex sphere coordinate. In the simple case of celestial sphere, it transforms as

$$z'^{1} = \tfrac{c+dz^{1}}{a+bz^{1}} \quad , \quad ad - bc = 1 \tag{14.21}$$

Then we see that in two sets of regular coordinates, the dependent variable $z^{\widetilde{1}}$ transforms under the chiral representation

$$z'^{\widetilde{1}} = \tfrac{\overline{c}+\overline{d}z^{\widetilde{1}}}{\overline{a}+\overline{b}z^{\widetilde{1}}} \quad , \quad ad - bc = 1 \tag{14.22}$$

This is implied by the incidence relation $z^{\widetilde{1}} - \overline{z^{1}} = O^{2}(\overline{z^{a}}, z^{\widetilde{0}})$, applied order by order.

The first remark is that the LCR-structure "sees" the chirality while the riemannian structure cannot. From the defining algebraic conditions (14.1) of a LCR-manifold, the first check of the chiral symmetry of a solitonic LCR-configuration is the irreducibility or reducibility of the hypersurface of $CP(3)$. We will see that the electron LCR-structure is chirally symmetric, while its neutrino LCR-structure is chirally asymmetric, because it is based on the reducible product of two planes of $CP(3)$.

14.4 Charge inversion

The gauge field is real and the action is also real. Therefore the complex conjugation interchanges only the complex vector $m \leftrightarrow \overline{m}$ of the tetrad. If the LCR-structure is realizable, i.e. it admits structure coordinates $(z^{\alpha}, z^{\widetilde{\beta}})$ such that

$$dz^{\alpha} = f_0^{\alpha}\ \ell_\mu dx^\mu + f_1^{\alpha}\ m_\mu dx^\mu \quad , \quad dz^{\widetilde{\alpha}} = f_{\widetilde{0}}^{\widetilde{\alpha}}\ n_\mu dx^\mu + f_{\widetilde{1}}^{\widetilde{\alpha}}\ \widetilde{m}_\mu dx^\mu$$
$$\ell = \ell_\alpha dz^\alpha\ ,\ m = m_\alpha dz^\alpha \quad , \quad n = n_{\widetilde{\alpha}} dz^{\widetilde{\alpha}}\ ,\ \widetilde{m} = \widetilde{m}_{\widetilde{\alpha}} dz^{\widetilde{\alpha}} \tag{14.23}$$

the above discrete transformation is equivalent to $(z'^{\alpha}, z'^{\widetilde{\beta}}) = (\overline{z^{\alpha}}, \overline{z^{\widetilde{\beta}}})$. We will call this discrete transformation, conjugate transformation and show that it is the particle↔antiparticle correspondence, observed in nature.

We already know that the LCR-structure defines the following class of symmetric and antisymmetric tensors

$$\begin{array}{l}
[g_{\mu\nu}] = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu \\
[J_{\mu\nu}] = \ell_\mu n_\nu - n_\mu \ell_\nu - m_\mu \overline{m}_\nu + \overline{m}_\mu m_\nu
\end{array} \tag{14.24}$$

A representative of the symmetric tensor provides the metric, which defines the conserved energy-momentum current. It is invariant under the conjugate transformation. Therefore conjugate solitonic configurations have the same masses. On the other hand, the representative of the antisymmetric tensors, which defines the conserved electromagnetic current, changes under the conjugate transformation. Therefore conjugate solitonic configuration have opposite charges, because the charge is defined as the integral over the surface $m \wedge \overline{m}$. These transformations will be extensively studied in the corresponding chapters.

15 AMBIENT KAEHLERIAN MANIFOLD

We have already seen that if a LCR-structure is realizable, it becomes a special totally real submanifold of a complex manifold. The embedding functions

$$\begin{array}{l}
\rho_{11}(\overline{z^{\alpha}}, z^{\alpha}) = 0 \quad , \quad \rho_{12}(\overline{z^{\alpha}}, z^{\widetilde{\alpha}}) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\alpha}}) = 0 \\
\frac{\partial \rho_{ij}}{\partial z^b} \neq 0 \neq \frac{\partial \rho_{ij}}{\partial \overline{z^b}}
\end{array} \tag{15.1}$$

of a general LCR-structure defines the following Kaehler metric and corresponding symplectic form

$$ds^2 = 2\frac{\partial^2 \det(\rho_{ij})}{\partial z^a \partial \overline{z^b}} dz^a d\overline{z^b} \quad , \quad \omega = 2i\frac{\partial^2 \det(\rho_{ij})}{\partial z^a \partial \overline{z^b}} dz^a \wedge d\overline{z^b} \tag{15.2}$$

The metric is generally indefinite, which becomes positive definite in a region of the ambient complex manifold. A straightforward calculation gives

$$\begin{array}{l}
\frac{\partial^2 \rho}{\partial z^a \partial \overline{z^b}} = 4[\rho_{22}\frac{\partial^2 \rho_{11}}{\partial z^a \partial \overline{z^b}} + \frac{\partial \rho_{11}}{\partial z^a}\frac{\partial \rho_{22}}{\partial \overline{z^b}} + \frac{\partial \rho_{22}}{\partial z^a}\frac{\partial \rho_{11}}{\partial \overline{z^b}} + \rho_{11}\frac{\partial^2 \rho_{22}}{\partial z^a \partial \overline{z^b}} - \\
\quad -\overline{\rho_{12}}\frac{\partial^2 \rho_{12}}{\partial z^a \partial \overline{z^b}} - \frac{\partial \overline{\rho_{12}}}{\partial z^a}\frac{\partial \rho_{12}}{\partial \overline{z^b}} - \frac{\partial \rho_{12}}{\partial z^a}\frac{\partial \overline{\rho_{12}}}{\partial \overline{z^b}} - \rho_{12}\frac{\partial^2 \overline{\rho_{12}}}{\partial z^a \partial \overline{z^b}}]
\end{array} \tag{15.3}$$

If we neglect the special dependence of the elements of the hermitian matrix ρ_{ij} on the left z^{β} and right $z^{\widetilde{\beta}}$ structure coordinates, we may diagonalize it with a unitary transformation

$$\begin{array}{l}(\rho_{ij}) = U^{\dagger}\begin{pmatrix}\rho_1 & 0 \\ 0 & \rho_2\end{pmatrix}U \\ \det(\rho_{ij}) = \rho_1\rho_2 \quad , \quad tr(\rho_{ij}) = \rho_1 + \rho_2\end{array} \tag{15.4}$$

Hence the real submanifold $\rho_{ij} = 0$, may take the form $\rho_1 = -\rho_2 = 0$. That is, the above special Kaehler metric is a Fefferman-like metric.

In the LCR-structure coordinates

$$z^I = (z^a; \overline{z^b}) = (z^{\alpha}, z^{\widetilde{\alpha}}; \overline{z^{\beta}}, \overline{z^{\widetilde{\beta}}}) \tag{15.5}$$

and on the surface ($\rho_{ij} = 0$) the metric takes the lorentzian form

$$\begin{array}{l}ds^2|_M = 4(\frac{\partial\rho_{11}}{\partial z^a}\frac{\partial\rho_{22}}{\partial\overline{z^b}} + \frac{\partial\rho_{22}}{\partial z^a}\frac{\partial\rho_{11}}{\partial\overline{z^b}} - \frac{\partial\overline{\rho_{12}}}{\partial z^a}\frac{\partial\rho_{12}}{\partial\overline{z^b}} - \frac{\partial\rho_{12}}{\partial z^a}\frac{\partial\overline{\rho_{12}}}{\partial\overline{z^b}})dz^a \otimes d\overline{z^b} = \\ \quad = 2(\ell\otimes n - m\otimes\overline{m})\end{array} \tag{15.6}$$

where the LCR-tetrad is found using the standard relations

$$\begin{array}{l}\ell = i(\partial - \overline{\partial})\rho_{11} \\ n = i(\partial - \overline{\partial})\rho_{22} \\ m = i(\partial - \overline{\partial})\overline{\rho_{12}} \\ \overline{m} = i(\partial - \overline{\partial})\rho_{12}\end{array} \tag{15.7}$$

The symplectic 2-form ω on the LCR-manifold vanishes

$$\begin{array}{l}\omega|_M = 4i(\frac{\partial\rho_{11}}{\partial z^a}\frac{\partial\rho_{22}}{\partial\overline{z^b}} + \frac{\partial\rho_{22}}{\partial z^a}\frac{\partial\rho_{11}}{\partial\overline{z^b}} - \frac{\partial\overline{\rho_{12}}}{\partial z^a}\frac{\partial\rho_{12}}{\partial\overline{z^b}} - \frac{\partial\rho_{12}}{\partial z^a}\frac{\partial\overline{\rho_{12}}}{\partial\overline{z^b}})dz^a \wedge d\overline{z^b} = \\ \quad = i(\ell\wedge n + n\wedge\ell - m\wedge\overline{m} - \overline{m}\wedge m) = 0\end{array} \tag{15.8}$$

Hence this submanifold is lagrangian relative to this class of symplectic forms and the induced metric is the compatible one with the corresponding LCR-structure. In fact I considered the precise Kaehler metric in order the LCR-manifold to become lagrangian submanifold of the ambient complex manifold. My ultimate goal is to use the above kaehlerian forms to apply geometric quantization with a polarization induced by the LCR-manifold.

Recall that the projectivization of the ambient complex manifold through the Kerr function implied that the structure coordinates are (generally meromorphic) functions of the projective coordinates r^a of the grassmannian space $G(4,2)$. Hence, after an holomorphic transformation, the Kaehler metric and corresponding symplectic 2-form become

$$ds^2 = \frac{\partial^2\rho'}{\partial r^a\partial\overline{r^b}}dr^a d\overline{r^b} \quad , \quad \omega = i\frac{\partial^2\rho'}{\partial r^a\partial\overline{r^b}}dr^a \wedge d\overline{r^b} \tag{15.9}$$

outside the possible singularities. The Kaehler potential is $\rho'(r,\overline{r}) = \rho(z(r), \overline{z}(\overline{r}))$. Using the definition $r^a := x^a + iy^a$ we find the following symplectic 2-form

$$\begin{array}{l}\omega = \frac{1}{4}(\frac{\partial^2\rho''}{\partial x^a\partial y^b} - \frac{\partial^2\rho''}{\partial x^b\partial y^a})(dx^a\wedge dx^b + dy^a\wedge dy^b) + \\ \quad + \frac{1}{2}(\frac{\partial^2\rho''}{\partial x^a\partial x^b} + \frac{\partial^2\rho''}{\partial y^a\partial y^b})dx^a\wedge dy^b\end{array} \tag{15.10}$$

where the Kaehler potential ρ'' is now function of x^a and y^b.

Example: We consider the (regular) degenerate LCR-structure. We find

$$
\begin{aligned}
&\rho_{11} = \tfrac{z^0 - \overline{z^0}}{i} = 0 \quad , \quad \rho_{12} = z^{\tilde{1}} - \overline{z^1} = 0 \quad , \quad \rho_{22} = \tfrac{z^{\tilde{0}} - \overline{z^{\tilde{0}}}}{i} = 0 \\
&\rho = -(z^0 - \overline{z^0})(z^{\tilde{0}} - \overline{z^{\tilde{0}}}) - (z^{\tilde{1}} - \overline{z^1})(\overline{z^{\tilde{1}}} - z^1) \\
&ds^2 = dz^0 d\overline{z^{\tilde{0}}} + dz^{\tilde{0}} d\overline{z^0} - dz^1 d\overline{z^1} - dz^{\tilde{1}} d\overline{z^{\tilde{1}}} \\
&\omega = i(dz^0 \wedge d\overline{z^{\tilde{0}}} + dz^{\tilde{0}} \wedge d\overline{z^0} - dz^1 \wedge d\overline{z^1} - dz^{\tilde{1}} \wedge d\overline{z^{\tilde{1}}})
\end{aligned}
\tag{15.11}
$$

One can easily see that the LCR-manifold is lagrangian. The holomorphic relations between the structure coordinates and the projective grassmannian coordinates are

$$
\begin{aligned}
&z^0 = \tfrac{r^0 - r^3}{2} \quad , \quad z^1 = \tfrac{r^1 + ir^2}{2} \quad , \quad z^{\tilde{0}} = \tfrac{r^0 + r^3}{2} \quad , \quad z^{\tilde{1}} = \tfrac{r^1 - ir^2}{2} \\
&\rho' = \tfrac{-1}{4}(r^a - \overline{r^a})(r^b - \overline{r^b})\eta_{ab} \quad , \quad \rho'' = -y^a y^b \eta_{ab}
\end{aligned}
\tag{15.12}
$$

$\square$.

Hence we see that the Kaehler potential $\rho = \det(\rho_{ij})$ makes the ambient complex manifold kaehlerian with its LCR-submanifold lagrangian. Recall that this is the framework to trigger the geometric quantization with polarization the LCR-submanifold.

15.1 Case of zero gravity LCR-manifolds

In classical mechanics the dynamical laws imply the trajectories of the particles. But in the case of flat (compatible with the Minkowski metric) LCR-manifolds the procedure seems to be inverted! A (generally complex) Newman trajectory completely determine the LCR-structure, its ambient kaehlerian manifold and hence its "dynamics". Recall that this ambient kaehlerian manifold is the grassmannian space $G(4,2)$ with the Kaehler potential ρ determined by the relations

$$
\begin{aligned}
&X^{nj} = \begin{pmatrix} \lambda^{Aj} \\ -ir_{A'B}\lambda^{Bj} \end{pmatrix} = \begin{pmatrix} \lambda^{Aj} \\ -i\xi^j_{A'B}(\tau_j)\lambda^{Bj} \end{pmatrix} \\
&z^0 = \tau_1 \ , \ z^1 = \tfrac{\lambda^{11}}{\lambda^{01}} \ , \ z^{\tilde{0}} = \tau_2 \ \ , \ z^{\tilde{1}} = -\tfrac{\lambda^{02}}{\lambda^{12}} \\
&\rho = \det(\rho_{ij}) = \det(X^\dagger E X)
\end{aligned}
\tag{15.13}
$$

The precise form of the Kaehler potential and the symplectic 2-form, as functions of the trajectory, is

$$
\begin{aligned}
\rho_{11} &= -i[(\xi^1_{0'0} - \overline{\xi^1_{0'0}}) + (\xi^1_{0'1} - \overline{\xi^1_{1'0}})z^1 - (\overline{\xi^1_{0'1}} - \xi^1_{1'0})\overline{z^1} + (\xi^1_{1'1} - \overline{\xi^1_{1'1}})z^1\overline{z^1}] \\
\rho_{12} &= -i[-\overline{\xi^1_{1'0}} + \xi^2_{0'1} + (\overline{\xi^1_{0'0}} - \xi^2_{0'0})z^{\tilde{1}} + (\xi^2_{1'1} - \overline{\xi^1_{1'1}})\overline{z^1} + (\overline{\xi^1_{0'1}} - \xi^2_{1'0})z^{\tilde{1}}\overline{z^1}] \\
\rho_{22} &= -i[(\xi^2_{1'1} - \overline{\xi^2_{1'1}}) + (\overline{\xi^2_{0'1}} - \xi^2_{1'0})z^{\tilde{1}} - (\xi^2_{0'1} - \overline{\xi^2_{1'0}})\overline{z^{\tilde{1}}} + (\xi^2_{0'0} - \overline{\xi^1_{0'0}})z^{\tilde{1}}\overline{z^{\tilde{1}}}]
\end{aligned}
\tag{15.14}
$$

15.2 Case of 2-d LCR-manifolds

After the recent negative experiments, where no supersymmetric particle has been detected, we may claim that the quite promising string theory is experimentally dead. But Polyakov action is a 2-dimensional toy mathematical model of the much more complicated 4-dimensional PCFT. Many mathematical features of 4-dimensional LCR-structure

$$
\begin{array}{l}
\rho_1(\overline{z^0}, z^0) = 0 \quad , \quad \rho_2(\overline{z^{\widetilde{0}}}, z^{\widetilde{0}}) = 0 \\
\frac{\partial \rho_i}{\partial z^b} \neq 0 \neq \frac{\partial \rho_i}{\partial \overline{z^b}}
\end{array}
\tag{15.15}
$$

appear in the 2-dimensional LCR-structure, which we could study to get computational ideas. One of theses is the two dimensional kaehlerian complex ambient manifold with Kaehler potential $K = \rho_1(\overline{z^0}, z^0)\rho_2(\overline{z^{\widetilde{0}}}, z^{\widetilde{0}})$, which has a lagrangian 2-dimensional LCR-submanifold. These toy manifolds deserve detailed study in order to understand the cobordism and/or the coincidence of the extrinsic approach as a Kaehler manifold with the intrinsic approach as a phase space.

16 LCR-MANIFOLDS IN BOUNDED DOMAIN

The identification of the natural Poincaré symmetry with the linear subgroup of the unbounded (Siegel) realization of the $SU(2,2)$ symmetric classical domain turned our investigation into this direction. But the bounded (Cartan) realization

$$
\begin{array}{l}
\begin{pmatrix} Y_1^\dagger & Y_2^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \succ 0 \quad \Longleftrightarrow \quad I - w^\dagger w \succ 0 \\
w := Y_2 Y_1^{-1}
\end{array}
\tag{16.1}
$$

is an holomorphic complete circular domain, which we will denote with D_B i.e. if $w \in D_B$, then $se^{i\varphi}w \in D_B$ with $\varphi \in [0, 2\pi)$ and $s \in [0,1)$. Its characteristic boundary (where the absolute value of every holomorphic function takes its maximum and vice-versa) is the group $U(2)$. It is a manifold, which admits LCR-structures determined by the conditions $\rho_{ij} = Y^\dagger E_B Y = 0$.

Hence, every point of $D_B \subset G(4,2)$ determines a line of $CP(3)$, which intersects an algebraic surface at a number of points equal to the degree of its Kerr polynomial. A pair of intersection points determine the following homogeneous coordinates

$$
Y^{ni} = \begin{pmatrix} \kappa_{ij}(w) \\ w_{kl}\kappa_{lj}(w) \end{pmatrix}
\tag{16.2}
$$

where w_{kl} are the projective coordinates and repeated indices usually indicate summation. $\kappa_{i1}(w)$ and $\kappa_{i2}(w)$ are the two sections, which projectively determine the corresponding sheets of the surface and respectively the analytic extension of the structure coordinates $z^a(w)$. They become the structure coordinates of the LCR-structure, when w becomes a point of the characteristic boundary $U(2)$.

The Cayley transformation

$$\begin{array}{l} r = i(I-w)(I+w)^{-1} = i(I+w)^{-1}(I-w) \\ \\ w = (iI-r)(iI+r)^{-1} = (iI+r)^{-1}(iI-r) \end{array} \tag{16.3}$$

restricted on the boundary becomes $U(2) \to \mathbb{R}^4$. This correspondence needs two sheets of $\mathbb{R}^4$ to become bijective, because the spinorial and vector representation $SU(2)/\{I,-I\} \leftrightarrow SO(3,\mathbb{R})$, are not bijective. It is easily seen regarding the forms of the $SU(2)$ and $SO(3,\mathbb{R})$ representations implied by exponentiation of their common Lie algebra

$$\begin{array}{l} SU(2): \quad U = e^{i\psi_j \frac{\sigma^j}{2}} = \cos\frac{\psi}{2} + i\widehat{\psi}_j \sigma^j \sin\frac{\psi}{2} \quad , \quad \psi_j =: \psi \widehat{\psi}_j \\ U(\widehat{\psi}_j, \psi + 4\pi) = U(\widehat{\psi}_j, \psi) = -U(\widehat{\psi}_j, \psi + 2\pi) \\ \\ SO(3,\mathbb{R}): \quad O = \delta_{ij}\cos\psi + \widehat{\psi}_i\widehat{\psi}_j(1-\cos\psi) + \epsilon_{ijk}\widehat{\psi}_k \sin\psi \quad , \quad \psi \in [0, 2\pi) \\ O(\widehat{\psi}_j, \psi + 2\pi) = O(\widehat{\psi}_j, \psi) \end{array} \tag{16.4}$$

Notice that the domain of the angle $\psi(>0)$ in $SU(2)$ is $[0, 4\pi)$, while $SO(3,\mathbb{R})$ is covered by $\psi \in [0, 2\pi)$. When $\psi =: 2\rho$ is in this domain (the reader should not confuse this angle ρ with the previous Kaehler potential), the cartesian coordinates of the $w = I$ chart

$$\begin{array}{l} x_+ = i(I-w)(I+w)^{-1} = i(I+w)^{-1}(I-w) \\ w^\dagger = w^{-1} \end{array} \tag{16.5}$$

is found assuming

$$\begin{array}{l} \widehat{\psi}_j = (-\sin\sigma\cos\chi,\ -\sin\sigma\sin\chi,\ \cos\sigma) \\ \\ w = e^{i\tau} \begin{pmatrix} \cos\rho + i\sin\rho\cos\sigma & -i\sin\rho\sin\sigma\ e^{-i\chi} \\ -i\sin\rho\sin\sigma\ e^{i\chi} & \cos\rho - i\sin\rho\cos\sigma \end{pmatrix} \\ \tau \in (-\pi, \pi) \quad , \quad \rho \in [0, 2\pi) \quad , \quad \sigma \in [0, \pi) \quad , \quad \chi \in (0, 2\pi) \end{array} \tag{16.6}$$

It has the form

$$\begin{array}{l} x_+^0 = \frac{\sin\tau}{\cos\tau + \cos\rho} \\ x_+^1 + ix_+^2 = \frac{\sin\rho}{\cos\tau+\cos\rho}\sin\sigma\ e^{i\chi} \\ x_+^3 = \frac{\sin\rho}{\cos\tau+\cos\rho}\cos\sigma \\ \tau \in (-\pi, \pi)\ ,\ \rho \in [0, \pi)\ ,\ \sigma \in [0, \pi)\ ,\ \chi \in (0, 2\pi) \\ \\ s := \frac{\sin\rho}{\cos\tau\ + \cos\rho} > 0 \quad \leftrightarrow \quad \cos\tau\ +\ \cos\rho > 0 \end{array} \tag{16.7}$$

The cartesian coordinates of the second $\mathbb{R}^4$-chart around the point $w = -I$ are

$$\begin{array}{l} x_- = i(I+w)(I-w)^{-1} = i(I-w)^{-1}(I+w) \\ w^\dagger = w^{-1} \end{array} \tag{16.8}$$

which have the form

$$\begin{array}{l}
x'^0 = \frac{\sin\tau}{\cos\tau - \cos\rho} \\
x'^1 + ix'^2 = -\frac{\sin\rho}{\cos\tau\,\cos\rho}\sin\sigma\ e^{i\chi} \\
x'^3 = \frac{\sin\rho}{\cos\tau - \cos\rho}\cos\sigma
\end{array} \tag{16.9}$$

Apparently the spherical angles σ, χ cannot be considered as the ordinary 3-dimensionl spherical angles. Therefore I prefer to cover the rest of $U(2)$ with

$$\begin{array}{l}
x_-^0 = \frac{\sin\tau}{\cos\tau + \cos\rho} \\
x_-^1 + ix_-^2 = -\frac{\sin\rho}{\cos\tau + \cos\rho}\sin\sigma\ e^{i\chi} \\
x_-^3 = -\frac{\sin\rho}{\cos\tau + \cos\rho}\cos\sigma \\
\tau \in (-\pi, \pi)\ ,\ \rho \in [0, \pi)\ ,\ \sigma \in [0, \pi)\ ,\ \chi \in (0, 2\pi) \\
\\
s := \frac{\sin\rho}{\cos\tau + \cos\rho} < 0 \quad \leftrightarrow \quad \cos\tau + \cos\rho < 0
\end{array} \tag{16.10}$$

Hence we conclude that the unbounded realization shows only the one sheet of the universe. As usual the angle parametrizations describe the entire bounded realization, with the problem of emergence of discontinuities at the boundaries of the angular variables.

In the Euler angle parametrization

$$\begin{array}{l}
SU(2) \ni U = \exp(-i\omega_k \frac{\sigma^k}{2}) = U_z(\gamma)U_y(\beta)U_z(\alpha) = e^{-i\gamma\frac{\sigma^3}{2}} e^{-i\beta\frac{\sigma^2}{2}} e^{-i\alpha\frac{\sigma^3}{2}} \\
\\
w = e^{i\tau} \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} \\
\tau \in (-\pi, \pi) \quad , \quad \alpha \in [0, 2\pi) \quad , \quad \beta \in [0, \pi] \quad , \quad \gamma \in [0, 2\pi)
\end{array} \tag{16.11}$$

the cartesian coordinates at $w = I$ have the form

$$\begin{array}{lll}
x_+^0 = \frac{\sin\tau}{\cos\tau + \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}} & , & x_+^1 = \frac{-\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}}{\cos\tau + \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}} \\
x_+^2 = \frac{-\sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2}}{\cos\tau + \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}} & , & x_+^3 = \frac{\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}}{\cos\tau + \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}}
\end{array} \tag{16.12}$$

Recall that the LCR-structure conditions $\rho_{ij} = 0$ are defined up to a factor function, which imply the tetrad-Weyl transformation. We may use it, in order to identify the Kaehler potential $K_B = \frac{1}{2}\det(I - w^\dagger w)$. For that we make the following successive steps

$$\begin{array}{l}
\kappa_{ij}(w) = \begin{pmatrix} 1 & \kappa_2(w) \\ \kappa_1(w) & 1 \end{pmatrix} \\
\det(\rho_{ij}) = \det Y^\dagger E_B Y = |\det\kappa|^2 \det(I - w^\dagger w) \\
K_B = \frac{\det(\rho_{ij})}{2|\det\kappa|^2} = \frac{1}{2}\det(I - w^\dagger w) \\
|\det\kappa| \neq 0
\end{array} \tag{16.13}$$

where the factor $1/2$ is introduced in order to find the ordinary metric of the $U(2)$ group. Recall that in the Kaehler potential of the Bergman metric intervenes the logarithm, which makes the metric singular at the boundary.

Identifying the ambient complex manifold with the bounded classical domain, the natural Kaehler metric takes the following form

$$\begin{array}{l} ds_B^2 = \frac{1}{2}[(-1+\overline{w_{22}}w_{22})dw_{11}d\overline{w_{11}} - \overline{w_{21}}w_{22}dw_{11}d\overline{w_{12}} - \overline{w_{12}}w_{22}dw_{11}d\overline{w_{21}} + \\ \quad +\overline{w_{11}}w_{22}dw_{11}d\overline{w_{22}} - \overline{w_{22}}w_{21}dw_{12}d\overline{w_{11}} + (-1+\overline{w_{21}}w_{21})dw_{12}d\overline{w_{12}} + \\ \quad +\overline{w_{12}}w_{21}dw_{12}d\overline{w_{21}} - \overline{w_{11}}w_{21}dw_{12}d\overline{w_{22}} - \overline{w_{22}}w_{12}dw_{21}d\overline{w_{11}} + \overline{w_{21}}w_{12}dw_{21}d\overline{w_{12}} + \\ \quad +(-1+\overline{w_{12}}w_{12})dw_{21}d\overline{w_{21}} - \overline{w_{11}}w_{12}dw_{21}d\overline{w_{22}} + \overline{w_{22}}w_{11}dw_{22}d\overline{w_{11}} - \\ \quad -\overline{w_{21}}w_{11}dw_{22}d\overline{w_{12}} - \overline{w_{12}}w_{11}dw_{22}d\overline{w_{21}} + (-1+\overline{w_{11}}w_{11})dw_{22}d\overline{w_{22}}] \end{array} \tag{16.14}$$

and the corresponding symplectic 2-form is easily implied. The induced metric on the characteristic boundary $U(2)$

$$\begin{array}{l} U = e^{i\tau}\begin{pmatrix} \cos\rho + i\sin\rho\cos\theta & -i\sin\rho\sin\theta\; e^{-i\varphi} \\ -i\sin\rho\sin\theta\; e^{i\varphi} & \cos\rho - i\sin\rho\cos\theta \end{pmatrix} \\ \tau\in(-\pi,\pi) \quad , \quad \rho\in[0,2\pi) \quad , \quad \theta\in[0,\pi) \quad , \quad \varphi\in[0,2\pi) \end{array} \tag{16.15}$$

is

$$ds_B^2 = (d\tau)^2 - (d\rho)^2 - \sin^2\rho(d\theta)^2 - \sin^2\rho\sin^2\theta(d\varphi)^2 \tag{16.16}$$

That is, the flat LCR-manifold and its ambient Kaehler manifold in the Cartan bounded realization have finite volumes.

16.1 Gravity emergence in the bounded realization

I have algebraically defined the gravitation as the deviation of the LCR-submanifold from the Shilov boundary of the $SU(2,2)$ symmetric domain. In the unbounded realization, it appears as the imaginary part $y^a = \frac{r^a - \overline{r^a}}{2i}$. In the bounded realization, the LCR-submanifold takes the form

$$\begin{array}{l} \rho_{ij}(\overline{Y^{mi}}, Y^{nj}) = \overline{Y^{mi}}E^{(B)}_{mn}Y^{nj} - G^{(B)}_{ij}(\overline{Y^{mi}}, Y^{nj}) = 0 \\ K_B(Y^{m1}) = 0 = K_B(Y^{m2}) \\ Y =: \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} =: \begin{pmatrix} Y_1 \\ wY_1 \end{pmatrix} \end{array} \tag{16.17}$$

Then we find

$$w^\dagger w = I - (Y_1^\dagger)^{-1}G^{(B)}(Y_1)^{-1} \tag{16.18}$$

Apparently the convenient parameterization of w is to separate its radial part $R = R^\dagger$ from its angular part $U = (U^\dagger)^{-1}$ i.e. $w =: UR$. Then we find

$$RR = I - (Y_1^\dagger)^{-1}G^{(B)}(Y_1)^{-1} \tag{16.19}$$

Notice that if gravity vanishes $G^{(B)} = 0$, we have $R = I$. In the case of non-vanishing gravity we have $R \neq I$ and it can be perturbatively computed as a function of the boundary coordinates at points where gravity is regular.

16.2 Affine transformations in the bounded realization

Recall that the physically observed Poincaré group is the affine (linear) subgroup which preserves the classical domain in its unbounded realization. There, we intuitively used the "black hole" metrics, which admit two geodetic and shear-free null congruences. The natural investigation is to find the physical role of the affine subgroup, which preserves the classical domain in its bounded realization.

The linear fractional transformations, which preserve the hermitian matrix E_B in the bounded realization have the form

$$\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$
$$w' = (A_{21} + A_{22}\ w)\,(A_{11} + A_{12}\ w)^{-1}$$

$$A_{11}^{\dagger} A_{11} - A_{21}^{\dagger} A_{21} = I \quad , \quad A_{11}^{\dagger} A_{12} - A_{21}^{\dagger} A_{22} = 0 \quad , \quad A_{22}^{\dagger} A_{22} - A_{12}^{\dagger} A_{12} = I \tag{16.20}$$

Its affine $A_{12} = 0$ subgroup is

$$\begin{gathered} \begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\ w' = A_{22}\ w A_{11}^{-1} \\ A_{11}^{\dagger} A_{11} = I \quad , \quad A_{22}^{\dagger} A_{22} = I \quad , \quad \det(A_{11} A_{22}) = 1 \end{gathered} \tag{16.21}$$

which coincides with the "infinity" affine subgroup $A_{21} = 0$. This group of affine transformations is the $S(U(2) \times U(2))$ subgroup of $SU(2,2)$. We will now look for the relation between this subgroup and the Poincaré×dilation subgroup. That is, what their common and non-common subgroups are? To check it, we have to find the unbounded form of the above transformations. It is

$$\begin{aligned} A &= \tfrac{1}{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \\ &= \tfrac{1}{2} \begin{pmatrix} A_{11} + A_{22} & A_{11} - A_{22} \\ A_{11} - A_{22} & A_{11} + A_{22} \end{pmatrix} \end{aligned} \tag{16.22}$$

Recall that the general form of the affine transformations in the unbounded realization (Poincaré transformations) is

$$\begin{gathered} \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} B & 0 \\ -iTB & (B^{\dagger})^{-1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ \det B \neq 0 \quad , \quad T^{\dagger} = T \end{gathered} \tag{16.23}$$

Apparently, the affine transformation of the bounded realization is affine in the unbounded realization if $A_{11} = A_{22} \in U(2)$. Hence it coincides with the corresponding (rotation) subgroup $(B^{\dagger})^{-1} = B$ of the Poincaré group. That is we find the trivial consequence, that the compact subgroup of the Poincaré subgroup of $SU(2,2)$ is subgroup of the maximal compact subgroup of $SU(2,2)$. Their non-common subgroup is the set of the $U(2)$ transformations

$$\begin{array}{l}
S = \frac{1}{2}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \\
\quad = \frac{1}{2}\begin{pmatrix} I+U & I-U \\ I-U & I+U \end{pmatrix} \\
U \in U(2)
\end{array} \tag{16.24}$$

which is the automorphism of the "flat universe" $U(2)$ through the transformations $w' = Uw$, as expected.

Under the above affine transformation, the bounded homogeneous coordinates of the boundary of the classical domain

$$\begin{array}{l}
Y^{nj} = \begin{pmatrix} \kappa_{ij} \\ w_{kl}\kappa_{lj} \end{pmatrix} = \begin{pmatrix} \kappa \\ w\kappa \end{pmatrix} \\
w^{\dagger} = w^{-1}
\end{array} \tag{16.25a}$$

transform as follows

$$\begin{array}{l}
\begin{pmatrix} \kappa' \\ w'\kappa' \end{pmatrix} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
\qquad\qquad \Downarrow \\
\kappa' = U_1\kappa \quad , \quad w' = U_2 w U_1^{\dagger} \\
U_1 \ , \ U_2 \in U(2)
\end{array} \tag{16.26a}$$

16.3 Symmetric bounded LCR-structures

The bounded affine group is

$$\begin{array}{l}
\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
U_1 \ , \ U_2 \in U(2) \quad , \quad \det(U_1 U_2) = 1
\end{array} \tag{16.27}$$

where U_1 and U_2 are independent $U(2)$ elements. In the unbounded realization it takes the form (16.22).

Hence the common subgroup of the affine groups of bounded and unbounded realizations is the rotation subgroup of the Poincaré group

$$\begin{array}{l}
\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
U \in SU(2)
\end{array} \tag{16.28}$$

The commuting infinitesimal transformations of the bounded affine non-common groups are

$$\begin{array}{lcl}\begin{pmatrix}I-i\varepsilon_1\frac{\sigma_3}{2} & 0\\ 0 & I\end{pmatrix} & \rightarrow & H_1=\begin{pmatrix}\frac{\sigma_3}{2} & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}I & 0\\ 0 & I-i\varepsilon_2\frac{\sigma_3}{2}\end{pmatrix} & \rightarrow & H_2=\begin{pmatrix}0 & 0\\ 0 & \frac{\sigma_3}{2}\end{pmatrix}\\ \begin{pmatrix}I-i\varepsilon_3 & 0\\ 0 & I+i\varepsilon_3\end{pmatrix} & \rightarrow & H_0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\end{array} \tag{16.29}$$

and the corresponding transformations of the (bounded) homogeneous coordinates are

$$\begin{array}{l}\delta Y^0=-\frac{i\varepsilon_1}{2}Y^0\ ,\quad \delta Y^1=\frac{i\varepsilon_1}{2}Y^1,\quad \delta Y^2=0,\quad \delta Y^3=0\\ \delta Y^0=0\ ,\quad \delta Y^1=0,\quad \delta Y^2=-\frac{i\varepsilon_2}{2}Y^2,\quad \delta Y^3=\frac{i\varepsilon_1}{2}Y^3\\ \delta Y^0=-i\varepsilon_3Y^0\ ,\quad \delta Y^1=-i\varepsilon_3Y^1,\quad \delta Y^2=i\varepsilon_3Y^2,\quad \delta Y^3=i\varepsilon_3Y^3\end{array} \tag{16.30}$$

The invariant quadratic Kerr polynomials are

$$\begin{gathered}K_1=A_{01}Y^0Y^1+A_{22}Y^2Y^2+A_{23}Y^2Y^3+A_{33}Y^3Y^3\\ K_2=A_{00}Y^0Y^0+A_{01}Y^0Y^1+A_{11}Y^1Y^1+A_{23}Y^2Y^3\\ K_3=A_{02}Y^0Y^2+A_{03}Y^0Y^3+A_{12}Y^1Y^2+A_{13}Y^1Y^3\end{gathered} \tag{16.31}$$

All the three Kerr polynomials can be regular.

The infinitesimal transformations of the diagonal generator of the common subgroup (z-rotation) is

$$\begin{gathered}\begin{pmatrix}I-i\varepsilon\frac{\sigma_3}{2} & 0\\ 0 & I-i\varepsilon\frac{\sigma_3}{2}\end{pmatrix}\quad \rightarrow H_1+H_2=\begin{pmatrix}\frac{\sigma_3}{2} & 0\\ 0 & \frac{\sigma_3}{2}\end{pmatrix}\\ \delta Y^0=-\frac{i\varepsilon}{2}Y^0\ ,\quad \delta Y^1=\frac{i\varepsilon}{2}Y^1,\quad \delta Y^2=-\frac{i\varepsilon}{2}Y^2,\quad \delta Y^3=\frac{i\varepsilon}{2}Y^3\end{gathered} \tag{16.32}$$

which implies the following invariant (non-degenerate) quadratic Kerr polynomial

$$K=A_{01}Y^0Y^1+A_{03}Y^0Y^3+A_{12}Y^1Y^2+A_{23}Y^2Y^3 \tag{16.33}$$

which is a transcription of the (12.15). The fact that the isomorphism z-rotation in the unbounded realization persists in the bounded realization makes this polynomial interesting. Notice that it has the same form with the unbounded one. Taking into account that the number of geodetic and shear-free null congruences of a "curved" LCR-manifold is limited to four. The implied quartic axially symmetric Kerr polynomial (12.25) into the present bounded realization has also the same form.

In the context of the G. Mack analysis of the "unitary representations of the conformal group $SU(2,2)$ with positive energy"[21], the set of the commuting generators in the bounded and the unbounded realizations is **NOT** the same. In the unbounded realization, the electron soliton is determined by its time-translation P^0 and z-rotation. In the present bounded realization, the appropriate automorphisms are $H_0 = \frac{1}{2}(P^0 + K^0)$ (which is essentially a τ translation) and z-rotation $H_1 + H_2$. Imposing the additional H_0 automorphism, the invariant quadratic Kerr polynomial is reduced to

$$K = A_{03} Y^0 Y^3 + A_{12} Y^1 Y^2 \tag{16.34}$$

and degenerate polynomials $Y^0 Y^1 = 0$ or $Y^2 Y^3 = 0$. Notice that if we try to impose an additional automorphism H_1 or H_2 to cover the entire maximal compact subgroup $S(U(2) \times U(2))$, the invariant polynomial becomes reducible with $A_{03} = 0$ or $A_{12} = 0$.

In the $Y^0 = 1 \Leftrightarrow (1, \kappa)^\top$ chart, the quadratic polynomial and its two solutions are

$$\begin{array}{l} w_{12}\kappa^2 + (w_{11} + c w_{22})\kappa + c w_{21} = 0 \\ \\ \kappa_{1(2)} = \frac{-(w_{11}+cw_{22}) \pm \sqrt{(w_{11}+cw_{22})^2 - 4cw_{12}w_{21}}}{2w_{12}} \end{array} \tag{16.35}$$

where $c \in \mathbb{C}$. In the Euler angle parametrization

$$\begin{array}{l} w = e^{i\frac{\delta}{2}} \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \\ \delta \in (-2\pi, 2\pi)\ ,\ \alpha \in (0, 2\pi)\ ,\ \beta \in (0, \pi)\ ,\ \gamma \in (0, 2\pi) \end{array} \tag{16.36}$$

and $c =: \cot\frac{\phi_1}{2} e^{i\phi_2}$ we find

$$\begin{array}{l} \kappa_{1(2)} = e^{i\frac{\gamma-\alpha+\phi_2}{2}} \frac{\cos\frac{\beta}{2}(\sin\frac{\alpha+\gamma+\phi_2+\phi_1}{2} + i \sin\frac{\alpha+\gamma+\phi_2-\phi_1}{2}) \pm \sqrt{\Delta}}{2\sin\frac{\phi_1}{2}\sin\frac{\beta}{2}} \\ \Delta = [\cos^2\frac{\beta}{2}(\sin^2\frac{\alpha+\gamma+\phi_2+\phi_1}{2} - \sin^2\frac{\alpha+\gamma+\phi_2-\phi_1}{2}) + 2\sin\phi_1 \sin^2\frac{\beta}{2}] + \\ \quad +2i\cos^2\frac{\beta}{2}\sin\frac{\alpha+\gamma+\phi_2+\phi_1}{2}\sin\frac{\alpha+\gamma+\phi_2-\phi_1}{2} \end{array} \tag{16.37}$$

and that the locus of the soliton, where the two solutions coincide, is

$$\begin{array}{llll} \sin\frac{\alpha+\gamma+\phi_2+\phi_1}{2} = 0 & , & \sin^2\frac{\alpha+\gamma+\phi_2-\phi_1}{2} & = 2\sin\phi_1\tan^2\frac{\beta}{2} \\ \mathit{or} & & & \\ \sin\frac{\alpha+\gamma+\phi_2-\phi_1}{2} = 0 & , & \sin^2\frac{\alpha+\gamma+\phi_2+\phi_1}{2} & = -2\sin\phi_1\tan^2\frac{\beta}{2} \end{array} \tag{16.38}$$

Considering the parametrization

$$Y^{ni} = \begin{pmatrix} 1 & -w^{\widetilde{1}} \\ w^1 & 1 \\ w^0 & c w^{\widetilde{0}} w^{\widetilde{1}} \\ \frac{-w^0 w^1}{c} & w^{\widetilde{0}} \end{pmatrix} \tag{16.39}$$

we can find the LCR-tetrad and the embedding conditions as usual.

16.4 Trajectories in bounded coordinates

Recall the definition of ruled surfaces (13.1)

$$\begin{aligned} Z^m(\tau, s) = (1-s)Z^{m1}(\tau) + sZ^{m2}(\tau) = \\ = Z^{m1}(\tau) + s[Z^{m2}(\tau) - Z^{m1}(\tau)] \end{aligned}$$

In the case of "flat" LCR-manifolds and the generating lines (rulings) correspond to complex points of the grassmannian manifold $G(4,2)$. In the unbounded realization of the grassmannian the rulings determine a trajectory $\xi(\tau) = \xi_a(\tau)\sigma^a$ which is essentially the Newman complex trajectory[26]. In the bounded realization the homogeneous coordinates of the ruled surfaces are

$$\begin{aligned} & Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ & Y_1 =: \begin{pmatrix} Z_1'^0 & Z_2'^0 \\ Z_1'^1 & Z_2'^1 \end{pmatrix} \quad , \quad Y_2 =: \begin{pmatrix} Z_1'^2 & Z_2'^2 \\ Z_1'^3 & Z_2'^3 \end{pmatrix} \end{aligned} \tag{16.40}$$

and the trajectory takes the form

$$q(\tau) =: Y_2 Y_1^{-1} =: \begin{pmatrix} q^0 - q^3 & -(q^1 - iq^2) \\ -(q^1 + iq^2) & q^0 + q^3 \end{pmatrix} = q_a \sigma^a \tag{16.41}$$

The Cayley relation between these two representations of the same trajectory is

$$\begin{aligned} & \xi = i(I-q)(I+q)^{-1} = i(I+q)^{-1}(I-q) \\ & q = (iI-\xi)(iI+\xi)^{-1} = (iI+\xi)^{-1}(iI-\xi) \end{aligned} \tag{16.42}$$

Recall that the developable surfaces, the massless (neutrinos) developable surfaces are defined by the condition

$$\det[Z_1^n, Z_2^n, \tfrac{dZ_1^n}{d\tau}, \tfrac{dZ_2^n}{d\tau}] \equiv 0 \tag{16.43}$$

From the apparent relation

$$\begin{aligned} & \det[\tfrac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}] = 1 \\ & \qquad\qquad \Downarrow \\ & \det[Z_1'^n, Z_2'^n, \tfrac{dZ_1'^n}{d\tau}, \tfrac{dZ_2'^n}{d\tau}] = \det[Z_1^n, Z_2^n, \tfrac{dZ_1^n}{d\tau}, \tfrac{dZ_2^n}{d\tau}] \equiv 0 \end{aligned} \tag{16.44}$$

and

$$\begin{aligned} & \det[Z_1'^n, Z_2'^n, \tfrac{dZ_1'^n}{d\tau}, \tfrac{dZ_2'^n}{d\tau}] = \det \begin{pmatrix} Y_1 & \dot{Y}_1 \\ qY_1 & \dot{q}Y_1 + q\dot{Y}_1 \end{pmatrix} = \\ & = \det[\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} Y_1 & \dot{Y}_1 \\ 0 & \dot{q}Y_1 \end{pmatrix}] = \det(\dot{q})(\det Y_1)^2 \end{aligned} \tag{16.45}$$

we find $\det(\dot{q}(\tau)) = 0$ too.

17 LCR TOPOLOGY

We have already pointed out that the relative invariants of the LCR-structure Φ_1, Φ_2 and Φ_3 turn out to be topological invariants, which we have first to check for precise LCR-tetrad in all the coordinate charts respecting the non-vanishing condition for the tetrad-Weyl factors.

The two real vectors $\ell^\mu\partial_\mu$ and $n^\mu\partial_\mu$ of the LCR-tetrad determine two integral curves x_ℓ^μ and x_n^μ respectively. Their tracing constitutes the mathematical tool to extend the LCR-manifold. Recall that the essential singularity, implied by the Hawking-Penrose theorems, is the clear indication of the limitations of the riemannian geometry. It is a general belief that these limitations will be solved by (unknown yet) quantum gravity. We will see, that in the context of PCFT it will completely solved even at the "classical level". I will show how the algebraic geometric origin of LCR-structure permit us to study and understand this limitation of general relativity. Therefore it is necessary to recast the journey from a hypersurface of $CP(3)$ to the real LCR-submanifold of the grassmannian space $G(4,2)$.

An algebraic hypersurface of $CP(3)$ is determined by a complex polynomial. If the polynomial is irreducible, it defines a regular hypersurface of $CP(3)$. If it is reducible to the product of two polynomials, it defines the union of two hypersurfaces. A line of $CP(3)$, that is a plane of $\mathbb{C}^4$, intersects the hypersurface to a number of points equal to the degree of the polynomial, called the degree of the algebraic surface. Choosing two X^{n1} and X^{n2} of these points we determine a point $X^{nj},\ j=1,2$ of the grassmannian space $G(4,2)$. This grassmannian space has an $SU(2,2)$ classical domain. There are two special coordinate charts. One where the classical domain is unbounded (not contained in the the affine space) with $\mathbb{R}^4$ its Shilov boundary and another one where the classical domain is bounded with $U(2)$ its Shilov boundary. In these two coordinate charts the relation between the homogeneous and projective coordinates are for the unbounded realization

$$X^{ni} = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ -i[(r^0-r^3)X^{01}-(r^1-ir^2)X^{11}] & -i[(r^0-r^3)X^{02}-(r^1-ir^2)X^{12}] \\ -i[-(r^1+ir^2)X^{01}+(r^0+r^3)X^{01}] & -i[-(r^1+ir^2)X^{02}+(r^0+r^3)X^{02}] \end{pmatrix} \tag{17.1}$$

with its inverted form

$$\begin{array}{l} r^0-r^3 = r_{0'0} = i\frac{X^{21}X^{12}-X^{11}X^{22}}{X^{01}X^{12}-X^{11}X^{02}} \\ -(r^1-ir^2) = r_{0'1} = i\frac{X^{01}X^{22}-X^{21}X^{02}}{X^{01}X^{12}-X^{11}X^{02}} \\ -(r^1+ir^2) = r_{1'0} = i\frac{X^{31}X^{12}-X^{11}X^{32}}{X^{01}X^{12}-X^{11}X^{02}} \\ r^0+r^3 = r_{1'1} = i\frac{X^{01}X^{32}-X^{31}X^{02}}{X^{01}X^{12}-X^{11}X^{02}} \end{array} \tag{17.2}$$

The LCR-structure conditions $\rho_{ij}(\overline{X^{ni}}, X^{mj})=0$ (notice the coincidence of the i,j indices) imply a projective computation of the fields $X^{mj}(x)$ and $r^a(x)$. The affine transformation, which preserves the boundary $\mathbb{R}^4$ of the classical domain is the Poincaré×dilation group regardless the fact that the complex variable $r^a = x^a + iy^a$ contains gravity $y^a(x)$. This physically means that the algebraic approach defines fields in representations of the Poincaré group, even in the case where gravity is present.

In the case of the coordinate chart Y^{ni}, where the classical domain (and its boundary $U(2)$) is bounded, completely analogous relations exist. Simply the affine transformation group changes from the Poincaré×dilation group to the $S[U(2) \times U(2)]$ group.

The left point X^{m1} of the hypersurface of $CP(3)$ determines the structure coordinates z^α and the pair (ℓ, m) of the LCR-tetrad. The second (right) point X^{m2} of the hypersurface of $CP(3)$ determines the structure coordinates $z^{\widetilde{\alpha}}$ and the pair $(n, \overline{m})$ of the LCR-tetrad. But we must be careful. The LCR conditions $\rho_{ij}(\overline{X^{ni}}, X^{mj}) = 0$ are **independently** homogeneous relative to X^{m1} and X^{m2} while the grassmannian manifold is homogeneous relative to the 2x2 matrices of X^{mi}. Only the "flat" LCR manifolds $X^\dagger E X = 0$ are properly defined in $G(4,2)$. Besides the ruled surfaces of $CP(3)$ determine a trajectory $\xi^a(\tau)$ of $G(4,2)$ and the LCR-structure needs a section $\lambda^{Ai}(\tau)$ to be fully defined. Therefore, I think the legitimate point of view is to consider the general LCR-manifold locally, as a local deformation of the Shilov boundary of classical domain. This apparently affects its compactness problem.

17.1 deRham cohomology

The $d^2 = 0$ property of the exterior derivative applied on smooth differential forms permits the existence of the following sequence

$$0 \to \Omega^0 \overset{d}{\to} \Omega^1 \overset{d}{\to} ...\Omega^k \overset{d}{\to} \Omega^{k+1} \overset{d}{\to} .... \tag{17.3}$$

where $image(d) \subseteq \ker(d)$, because not all closed smooth forms are exact. Ω^k is the set of k-degree differential forms of a manifold M. Hence the quotient group $H^k(M) := \ker(d)/image(d)$ is not always empty and it defines the deRham cohomology.

The Poincaré lemma states that all the k-cohomologies of an open connected set (and $\mathbb{R}^n$) are empty but zero

$$\begin{array}{l} H^0(\mathbb{R}^n) = \mathbb{R} \\ H^k(\mathbb{R}^n) = 0, \forall\ k \neq 0 \end{array} \tag{17.4}$$

because constant functions are closed, while they cannot be exact.

The cohomology invariants may change if the set of functions is changed. If the set of "smooth forms" is replaced by "smooth forms with compact support" we have

$$\begin{array}{l} H^0_c(\mathbb{R}^n) = 0 \\ H^k_c(\mathbb{R}^n) = 0, \forall\ k \neq 0, n \\ H^n_c(\mathbb{R}^n) = \mathbb{R} \end{array} \tag{17.5}$$

$H^0_c(\mathbb{R}^n) = 0$ means that there are no constant functions with compact support and $H^n_c(\mathbb{R}^n) = \mathbb{R}$ means that there are n-forms which are not exact.

If the set of "smooth forms" is replaced with the set of deRham currents (forms with coefficients distributions) the cohomology invariants depend on the kind of distributions. The deRham cohomology on distributions based on test functions with compact support is equivalent to the deRham cohomology $H_c^k(\mathbb{R}^n)$ on smooth functions with compact support (C^∞). Then the (closed) n-form which is not exact is

$$f = c\delta(x)dx^1 \wedge ... \wedge dx^n \quad c \in \mathbb{R} \tag{17.6}$$

On the other hand the ordinary (on smooth functions) deRham cohomology $H^k(\mathbb{R}^n)$ is equivalent to the deRham cohomology on distributions with compact support, which are based on test smooth functions (C^∞). These properties are derived by simply applying the definition of the derivative of the generalized function. Consider the case of the Coulomb field $F = d(A_0 dt)$. As a distribution it is a well defined exact form.

A realizable LCR-structure defines an ambient complex manifold with a LCR-covariant Kaehler metric, admitting as polarization the LCR-manifold. In this Kaehler manifold, the Dolbeault cohomology may be defined as usual.

Part III

SOLITONIC LEPTONS AND QUARKS

Synopsis

In this chapter we study the differences between the trivial "light-cone" LCR-structure and the flatprint electron LCR-structure viewed as structures of the bounded and unbounded realization of the $U(2)$ boundary of the classical domain. The geometry is studied via the tracing of the integral curves of the real vectors ℓ^μ and n^μ of the LCR-tetrad. It becomes clear how the naked singularity of the electron creates the ring-hole communicating the $r > 0$ and $r < 0$ regions of $U(2)$, which does not exist in the "light-cone" and "Schwartzschild" LCR-structures. The gravitational and electromagnetic "dressings" of the electron soliton define its energy-momentum, angular momentum and charge permitting us to identify the positron. A precise Cartan lift of the LCR-manifold implies the electroweak U(2)-connection (gauge potential), which is a rearrangement of the geodetic and shear-free null tetrad of the corresponding Einstein metric. The relative invariants are related with the Higgs field. The electronic neutrino is identified with the LCR-structure implied by the massless (developable) ruled surface companion of the electron (massive) ruled surface of $CP(3)$. I identify the leptonic number as the $\frac{-a}{|a|}$ Hopf invariant (linking number) of the left chiral part of the electron (and its neutrino). This suggests to identify the other leptonic generations with the ruled surfaces with higher Hopf invariants. I think that the limited number of leptonic generations could be related with the up to four geodetic and shear-free congruences of their gravitational dressings.

After a careful analysis of the gauge field-like equations in a LCR-manifold background, I find null and non-null colored abelian solutions with distributional charges. They are explicitly computed in the static electron LCR-manifold. I identify the non-null solution with the quark corresponding to the electron family. A precise Cartan lift of the Kerr surface of $CP(3)$ provides a $SU(3)$ connction, which seems to be very promising.

The efforts of Einstein to extend general relativity to include the other interactions are well known. The appearance of the above stabled leptonic and colored distributional solitons by the simple consideration of the LCR-structure as the fundamental structure in the place of riemannian structure of general relativity indicates that PCFT may be the theory that Einstein was looking for. Besides, the Hawking-Penrose singularity theorems are bypassed even at the classical level and they should be viewed as drawbacks of riemannian geometry.

18 "FLAT" LCR-MANIFOLDS

The fundamental notion of pseudo-conformal field theory (PCFT) is the lorentzian CR-structure (LCR-structure). The leptons are distributional solitonic LCR-manifolds. The distributional character of the solitons is a fundamental ingredient of the LCR-structure. Let me clarify it once more. In order to apply the (holomorphic) Frobenius theorem, we have to complexify spacetime. Then we find that the structure coordinates $(z^{\alpha}, z^{\widetilde{\beta}})$ satisfy special conditions, which determine a real 4-dimensional submanifold. LCR-structure is a special totally real CR-structure. So, in a neighborhood of a regular point there are locally analytic transformations, which give the LCR-conditions ρ_{ij} the simple form $\frac{r^b - \overline{r^b}}{2i} = 0$. But generally, the analytic transformations cannot be extended over the entire ambient space. At the non-analytic compact surfaces (they cannot be isolated points because of Hartog's theorem) will appear the singular region of the generalized function. Recall that a generalized function is described by a locally integrable singular function, which is the potential of the ladder of the distributional derivatives. The starting point is the LCR-manifold

$$\begin{aligned}
&\rho_{11}(\overline{z^{\alpha}}, z^{\alpha}) = 0 \quad , \quad \rho_{12}\left(\overline{z^{\alpha}}, z^{\widetilde{\alpha}}\right) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\alpha}}) = 0 \\
&\frac{\partial \rho_{ij}}{\partial z^b} \neq 0 \neq \frac{\partial \rho_{ij}}{\partial \overline{z^b}}
\end{aligned} \tag{18.1}$$

which determines a local coframe with its normal $d\rho_{ij}$ and tangent 1-forms

$$\begin{aligned}
&\ell = i(\partial - \overline{\partial})\rho_{11} \quad , \quad n = i(\partial - \overline{\partial})\rho_{22} \quad , \quad m = i(\partial - \overline{\partial})\overline{\rho_{12}} \\
&\begin{pmatrix} \ell & \overline{m} \\ m & n \end{pmatrix} = i(\partial - \overline{\partial}) \begin{pmatrix} \rho_{11} & \rho_{12} \\ \overline{\rho_{12}} & \rho_{22} \end{pmatrix}
\end{aligned} \tag{18.2}$$

The algebraic study of LCR-structure is based on 2-dimensional algebraic surfaces of $CP(3)$. Two points $X^{ni}, i = 1, 2$ of the surface determine a line in $CP(3)$ and a point

$$X^{ni} = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ X^{21} & X^{22} \\ X^{31} & X^{32} \end{pmatrix} = \begin{pmatrix} \lambda \\ -ir\lambda \end{pmatrix} \tag{18.3}$$

in the grassmannian space $G(4, 2)$, where X^{ni} are the homogeneous coordinates of the point and the four complex variables of the 2×2 matrix elements of r are its projective coordinates in the chart $(\det \lambda \neq 0)$. The LCR-structure is algebraically determined by the relations (10.8)

$$\begin{aligned}
&\rho_{11}(\overline{X^{m1}}, X^{n1}) = 0 \quad , \quad \rho_{12}(\overline{X^{m1}}, X^{n2}) = 0 \quad , \quad \rho_{22}(\overline{X^{m2}}, X^{n2}) = 0 \\
&K(X^{mi}) = 0
\end{aligned} \tag{18.4}$$

where $\rho_{11}(\cdot,\cdot)$, $\rho_{22}(\cdot,\cdot)$ are real homogeneous functions, $\rho_{12}(\cdot,\cdot)$ is complex and $K(\cdot)$ is the homogeneous holomorphic function (Kerr function) of the algebraic surface. The LCR-structures with $\rho_{ij} := \overline{X^{ni}} E_{nm} X^{mj}$ with E_{nm} a $SU(2,2)$ symmetric matrix, are called "flat", because their class of metrics $[g_{\mu\nu}]$ contains the Minkowski metric. Notice the essential difference between the present "flat" spacetime and the flat spacetime of general relativity, which coincides with $\mathbb{R}^4$ endowed with the Minkowski metric. Because of the spinorial representation of the Poincaré group, the present "flat" spacetime is twice the flat spacetime of general relativity. The purpose of the present section is to clarify this first essential difference of PCFT with general relativity.

The search for appropriate "flat"-manifolds coincides with the well defined LCR-structures in the characteristic boundary of the $SU(2,2)$ symmetric classical domain. There are two special realizations of the domain, the unbounded (Siegel) one with E_U and the bounded (Cartan) one with E_B, where

$$\begin{array}{l}
E_U := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad , \quad E_B := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}
\end{array} \tag{18.5}$$

The relation between the homogeneous and projective coordinates $X,\ r$ of the unbounded realization, with the corresponding coordinates $Y,\ w$ of the bounded realization are

$$\begin{array}{l}
X = \begin{pmatrix} X_1 \\ -irX_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}\begin{pmatrix} Y_1 \\ wY_1 \end{pmatrix} \\
r = i(I-w)(I+w)^{-1} = i(I+w)^{-1}(I-w)
\end{array} \tag{18.6}$$

The points at the boundary satisfy the conditions $r = r^\dagger$ and $w\dagger w = I$, that is, they have the parametrizations

$$\begin{array}{l}
r = x_a\sigma^a = \begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} \\
w = e^{i\tau}\begin{pmatrix} \cos\rho + i\sin\rho\cos\sigma & -i\sin\rho\sin\sigma\ e^{-i\chi} \\ -i\sin\rho\sin\sigma\ e^{i\chi} & \cos\rho - i\sin\rho\cos\sigma \end{pmatrix} \\
\tau \in (-\pi,\pi)\ ,\ \rho \in [0,2\pi)\ ,\ \sigma \in [0,\pi]\ ,\ \chi \in [0,2\pi)
\end{array} \tag{18.7}$$

Notice that $U(2) \to \mathbb{R}^4$ Cayley transformation is $2 \Longleftrightarrow 1$ with

$$\begin{array}{l}
For\ s := R_0\frac{\sin\rho}{\cos\tau\ +\cos\rho} > 0 \\
x^0 = T_0\frac{\sin\tau}{\cos\tau\ +\ \cos\rho} \\
x^1 + ix^2 = R_0\frac{\sin\rho}{\cos\tau\ +\cos\rho}\sin\sigma\ e^{i\chi} \\
x^3 = R_0\frac{\sin\rho}{\cos\tau+\cos\rho}\cos\sigma \\
\tau \in (0,2\pi)\ ,\ \rho \in [0,\pi)\ ,\ \sigma \in [0,\pi)\ ,\ \chi \in (0,2\pi) \\
Jacobian = -\frac{\sin^2\rho\sin\sigma}{(\cos\tau\ +\ \cos\rho)^4}
\end{array} \tag{18.8}$$

the one $\mathbb{R}^4$ sheet. The Cayley transformation centered around the point $w = -I$ is

$$x_- = i(I+w)(I-w)^{-1} = i(I-w)^{-1}(I+w)$$

$$\begin{aligned}
&x_-^0 = \frac{\sin\tau}{\cos\tau \ - \ \cos\rho} \\
&x_-^1 + ix_-^2 = -\frac{\sin\rho}{\cos\tau \ -\cos\rho}\sin\sigma \ e^{i\chi} \\
&x_-^3 = \frac{\sin\rho}{\cos\tau \ - \ \cos\rho}\cos\sigma
\end{aligned} \tag{18.9}$$

Below I will prefer to take the second $\mathbb{R}^4$ to be $s < 0$. The constants T_0 and R_0 are related to the S^1 and $SU(2) = S^3$ sizes and they are usually assumed to be equal to one. Notice that, if we identify σ, χ with ordinary spherical angles, this transformation coincides with the artificial Penrose compactification. But now it **is not artificial**. The bounded "flat" LCR-manifold is the union of two compactified Minkowski spacetimes, which communicate through the Penrose scri+ and scri- boundaries.

The importance of the "flat" manifolds comes from their proper embedding in the grassmannian manifold. Notice that the general form (18.4) does not properly define a real submanifold of $G(4,2)$. Their bounded realization permit us to have a global view of the solution, because it belongs to one projective coordinate system. Their unbounded realization hides singularities at "infinity", but it permit us better understand how the conservation laws of charge and energy-momentum fix the tetrad-Weyl symmetry.

18.1 "Natural $U(2)$" LCR-structure

I will usually denote a general point in $CP(3)$ with Z^n, with X^{ni} a point in $G(4,2)$ with unbounded boundary of classical domain $\mathbb{R}^4$ and with Y^{ni} a point in $G(4,2)$ with bounded boundary $U(2)$ of the classical domain. The first example of "flat" LCR-manifold is

$$\overline{Y^{ni}}E_{nm}^{(B)}Y^{mj} = 0 \quad , \quad Y^0Y^1 = 0 \tag{18.10}$$

which is well defined in the well defined group manifold $U(2)$. The proper regular grassmannian coordinates are

$$\begin{aligned}
&Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ w^0 & w^{\widetilde{1}} \\ w^1 & w^{\widetilde{0}} \end{pmatrix} = \begin{pmatrix} I \\ w \end{pmatrix} \\
&I - w^\dagger w = 0
\end{aligned} \tag{18.11}$$

Then we find the "natural $U(2)$" LCR-structure embedding conditions

$$\begin{aligned}
&\rho_{11} = w^0\overline{w^0} + w^1\overline{w^1} - 1 = 0 \\
&\rho_{12} = \overline{w^0}w^{\widetilde{1}} + w^{\widetilde{0}}\overline{w^1} = 0 \\
&\rho_{22} = w^{\widetilde{0}}\overline{w^{\widetilde{0}}} + w^{\widetilde{1}}\overline{w^{\widetilde{1}}} - 1 = 0
\end{aligned} \tag{18.12}$$

We already know that this LCR-structure is not degenerate. In the ordinary $U(2)$ parametrization, the structure coordinates are

$$\begin{array}{l} w = e^{i\tau} \begin{pmatrix} \cos\rho + i\sin\rho\cos\theta & -i\sin\rho\sin\theta \; e^{-i\varphi} \\ -i\sin\rho\sin\theta \; e^{i\varphi} & \cos\rho - i\sin\rho\cos\theta \end{pmatrix} \\ \tau \in (0, 2\pi) \quad , \quad \rho \in [0, 2\pi) \quad , \quad \theta \in [0, \pi) \quad , \quad \varphi \in (0, 2\pi) \end{array} \tag{18.13}$$

and the LCR-tetrad with the corresponding class of metrics is

$$\begin{array}{l} e = -iw^{-1}dw =: \begin{pmatrix} \ell & \overline{m} \\ m & n \end{pmatrix} \quad , \quad de - ie \wedge e = 0 \\ \\ d\ell = im \wedge \overline{m} \quad , \quad dn = -im \wedge \overline{m} \quad , \quad dm = i(\ell - n) \wedge m \\ [g] = \Lambda N(\ell_\mu n_\nu + \ell_\nu n_\mu) - M\overline{M}(m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu) dx^\mu dx^\nu = \\ \qquad = [(d\tau)^2 - (d\rho)^2 - \sin^2\rho (d\theta)^2 - \sin^2\rho \sin^2\theta (d\varphi)^2] \end{array} \tag{18.14}$$

The reader should notice that it is **NOT** a degenerate LCR-structure, i.e. its relative invariants Φ_1 and Φ_2 do not vanish.

The Euler angle parametrization

$$\begin{array}{l} w = e^{i\tau} \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \\ \tau \in (0, 2\pi) \quad , \quad \alpha \in [0, 4\pi) \quad , \quad \beta \in [0, \pi] \quad , \quad \gamma \in [0, 2\pi) \end{array} \tag{18.15}$$

"implies" (up to an accomodated sign) the Taub-NUT LCR-structure (4.27-4.29) through the following coordinate identifications

$$\begin{array}{l} \tau = \frac{t}{4l} \quad , \quad \alpha = \frac{r'}{2l} - \varphi \quad , \quad \beta = \theta \quad , \quad \gamma = \varphi \\ \\ z'^0 = e^{i\frac{t-r'}{4l}} \cos\frac{\theta}{2} = w^0 \quad , \quad z'^1 = \frac{-w^1}{w^0} = e^{i\varphi} \tan\frac{\theta}{2} \\ z'^{\widetilde{0}} = e^{i\frac{t+r'}{4l}} \cos\frac{\theta}{2} = w^{\widetilde{0}} \quad , \quad z'^{\widetilde{1}} = \frac{w^{\widetilde{1}}}{w^{\widetilde{0}}} = e^{-i\varphi} \tan\frac{\theta}{2} \end{array} \tag{18.16}$$

Hence the "Taub-NUT" LCR-structure is equivalent to the "natural $U(2)$" LCR-structure and the Taub-NUT parameter l is a scale parameter.

It is interesting to look at the flow $w_\ell(\tau)$ of the ℓ LCR-rays in the bounded $U(2) = S^1 \times S^3$ coordinates, using the S^1 parameter τ as affine parameter. It has the form

$$w_\ell(\tau) = \begin{pmatrix} c^0 & -\overline{c^1} e^{2i\tau} \\ c^1 & \overline{c^0} e^{2i\tau} \end{pmatrix} \tag{18.17}$$

where c^0 and c^1 are constants. Recall that the structure coordinates w^0 and w^1 are constants along these rays because $\ell^\mu \partial_\mu w^0 = 0 = \ell^\mu \partial_\mu w^1$. The corresponding flow $w_n(\tau)$ of the n LCR-rays is

$$w_n(\tau) = \begin{pmatrix} \overline{c^{\widetilde{0}}} e^{2i\tau} & c^{\widetilde{1}} \\ -\overline{c^{\widetilde{1}}} e^{2i\tau} & c^{\widetilde{0}} \end{pmatrix} \tag{18.18}$$

where $c^{\widetilde{0}}$ and $c^{\widetilde{1}}$ are constants.

In the unbounded realization, the Kerr function Y^0Y^1 of the "natural $U(2)$" LCR-structure takes the form $K(X) = (X^0 + X^2)(X^1 + X^3)$ and the LCR-conditions

$$\begin{aligned} & X = \begin{pmatrix} 1 & -z^{\widetilde{1}} \\ z^1 & 1 \\ -i(z^0+i) & z^{\widetilde{1}} \\ -z^1 & -i(z^{\widetilde{0}}+i) \end{pmatrix} = \begin{pmatrix} \lambda \\ -ix\lambda \end{pmatrix} \\ & X^\dagger E^{(U)} X = 0 \end{aligned} \tag{18.19}$$

are satisfied with $x^\dagger = x$ and the explicit forms of the structure coordinates are

$$\begin{array}{ll} z^1 = \frac{x^1+ix^2}{i+x^0+x^3} \quad , & z^0 = x^0 - x^3 - i - \frac{(x^1)^2+(x^2)^2}{i+x^0+x^3} \\ z^{\widetilde{1}} = -\frac{x^1-ix^2}{i+x^0-x^3} \quad , & z^{\widetilde{0}} = x^0 + x^3 - i - \frac{(x^1)^2+(x^2)^2}{i+x^0-x^3} \end{array}$$

$$z^0 - \overline{z^0} + 2i(1 - z^1\overline{z^1}) = 0 \quad , \quad z^{\widetilde{0}} - \overline{z^{\widetilde{0}}} + 2i(1 - z^{\widetilde{1}}\overline{z^{\widetilde{1}}}) = 0 \quad , \quad z^{\widetilde{1}}\overline{z^0} + z^{\widetilde{0}}\overline{z^1} = 0 \tag{18.20}$$

There is no singularity in $\mathbb{R}^4$, because $\det\lambda \neq 0$. We should expect it, because this LCR-structure does not have any singularity in the proper projective coordinate system, where the entire LCR-manifold belongs.

The "natural $U(2)$" LCR-structure may be viewed as a link. The point is to consider the one real condition of (18.12) as an evolution of the other real condition and the evolution rule determined by the complex condition. In the case of the electron and neutrino LCR-structures, we will relate these links with the different leptonic generations. We consider the two closed corresponding loops

$$\begin{aligned} & w'^0 = e^{2im\pi s} w^0 \quad , \quad w'^1 = e^{2in\pi s} w^1 \\ & m, n = coprime\ integers \end{aligned} \tag{18.21}$$

which preserves the first real condition and $\frac{n}{m}$ is their relative homotopy. Then the complex condition implies

$$\frac{w^{\widetilde{1}}}{w^{\widetilde{0}}} = -e^{2i\frac{n}{m}\pi s} \frac{w^1}{w^0} \tag{18.22}$$

which means that the evolution preserves the topology of links i.e. they have the same linking number, which is a topological condition for a smooth LCR-structure.

18.2 "Cartesian light-cone" LCR-structure

In order to clarify the above picture, we start with the very simple "cartesian light-cone" LCR-structure determined by the quadratic polynomial $X^0X^1 = 0$, which correspond to the union of the two planes $X^0 = 0 = X^1$. Taking one point at each polynomial, we find the grassmannian region

$$X^{ni} = \begin{pmatrix} X^{01} & 0 \\ 0 & X^{12} \\ X^{21} & X^{22} \\ X^{31} & X^{32} \end{pmatrix} =: \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{18.23}$$

which must be a rank-2 matrix, otherwise it is not a point of the grassmannian space. The branch curve is $Z = (0, 0, Z^2, Z^3)^\top$. Let us first consider the following simple chart $\det X_1 \neq 0$ and the corresponding projective coordinates and structure coordinates of the flat LCR-structure

$$\begin{array}{l} \det X_1 \neq 0 \quad , \quad x = iX_2(X_1)^{-1} \\ z^0 := i\frac{X^{21}}{X^{01}} = x^0 - x^3 \quad , \quad z^1 := -i\frac{X^{31}}{X^{01}} = x^1 + ix^2 \\ z^{\widetilde{0}} := i\frac{X^{32}}{X^{12}} = x^0 + x^3 \quad , \quad z^{\widetilde{1}} := -i\frac{X^{22}}{X^{12}} = \overline{z^1} \\ \\ \ell_\mu dx^\mu = \Lambda dz^0 \quad , \quad m_\mu dx^\mu = M dz^1 \\ n_\mu dx^\mu = \Lambda dz^{\widetilde{0}} \quad , \quad \overline{m}_\mu dx^\mu = \overline{M} dz^{\widetilde{1}} \end{array} \tag{18.24}$$

The integral curves $x_\ell^\mu(\sigma)$ of $\ell^\mu \partial_\mu$ are found using the definition of the projective coordinates

$$\begin{array}{ll} x_\ell^0 = i\frac{(X^{01}X^{32} - X^{31}X^{02}) + (X^{21}X^{12} - X^{11}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})} & = \frac{z^{\widetilde{0}} + z^0}{2} = \kappa \\ x_\ell^1 = i\frac{(X^{11}X^{32} - X^{31}X^{12}) + (X^{21}X^{02} - X^{01}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})} & = \frac{z^1 + z^{\widetilde{1}}}{2} = \frac{z^1 + \overline{z^1}}{2} \\ x_\ell^2 = i\frac{(X^{11}X^{32} - X^{31}X^{12}) - (X^{21}X^{02} - X^{01}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})} & = \frac{z^1 - z^{\widetilde{1}}}{2i} = \frac{z^1 - \overline{z^1}}{2i} \\ x_\ell^3 = i\frac{(X^{01}X^{32} - X^{31}X^{02}) - (X^{21}X^{12} - X^{11}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})} & = \frac{z^{\widetilde{0}} - z^0}{2} = \kappa - z^0 \end{array} \tag{18.25}$$

and the fact that z^0 and z^1 are constant along the curves, because $\ell^\mu \partial_\mu z^0 = 0 = \ell^\mu \partial_\mu z^1$. The integral curves $x_n^\mu(\sigma')$ of $n^\mu \partial_\mu$ are also found to be

$$\begin{array}{l} x_n^0 = \frac{z^{\widetilde{0}} + z^0}{2} = \kappa' \\ x_n^1 = \frac{z^1 + z^{\widetilde{1}}}{2} = \frac{\overline{z^{\widetilde{1}}} + z^{\widetilde{1}}}{2} \\ x_n^2 = \frac{z^1 - z^{\widetilde{1}}}{2i} = \frac{\overline{z^{\widetilde{1}}} - z^{\widetilde{1}}}{2i} \\ x_n^3 = \frac{z^{\widetilde{0}} - z^0}{2} = z^{\widetilde{0}} - \kappa' \end{array} \tag{18.26}$$

and the fact that $z^{\widetilde{0}}$ and $z^{\widetilde{1}}$ are constant along the curves, because $n^\mu \partial_\mu z^{\widetilde{0}} = 0 = n^\mu \partial_\mu z^{\widetilde{1}}$. Notice that no singularity appears neither to the tetrad, nor the structure coordinates and the integral curves. This should be expected, because the branch curve does not belong to this patch.

Let us now consider the chart $\det X_2 \neq 0$ and the corresponding projective coordinates and structure coordinates of the flat LCR-structure

$$\det X_2 \neq 0 \quad , \quad x' := -iX_1(X_2)^{-1} = x^{-1} \tag{18.27}$$

There is no need to go again through all the above procedure. In order to find the LCR-singularity, we write the tetrad in the new coordinate system and choose tetrad-Weyl factors such that the tetrad vectors do not have singularities. Then the possible physical singularity will emerge as the region where the tetrad is not linear independent. So we start from the tetrad

$$\begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} = \frac{1}{\eta_{ab}x'^a x'^b} \begin{pmatrix} x'^0 + x'^3 & (x'^1 - ix'^2) \\ (x'^1 + ix'^2) & x'^0 - x'^3 \end{pmatrix}$$

$$\begin{array}{l} \ell' = (\eta_{ab}x'^a x'^b)^2 [d\frac{x'^0}{\eta_{ab}x'^a x'^b} + d\frac{x'^3}{\eta_{ab}x'^a x'^b}] \\ n' = (\eta_{ab}x'^a x'^b)^2 [d\frac{x'^0}{\eta_{ab}x'^a x'^b} - d\frac{x'^3}{\eta_{ab}x'^a x'^b}] \\ m' = -(\eta_{ab}x'^a x'^b)^2 [d\frac{x'^1}{\eta_{ab}x'^a x'^b} + id\frac{x'^2}{\eta_{ab}x'^a x'^b}] \end{array} \tag{18.28}$$

Then we have

$$\ell' \wedge m' \wedge n' \wedge \overline{m}' = 4i(\eta_{ab}x'^a x'^b)^4 dx'^0 \wedge dx'^1 \wedge dx'^2 \wedge dx'^3 \tag{18.29}$$

which indicates that the singularity is at $(\eta_{ab}x'^a x'^b) = 0$ in the new coordinate system.

From the above appearance of the singularity, it becomes clear that the first coordinate system hides the singularity of the "cartesian light-cone" LCR-structure by sending it to infinity. Therefore, in order to have a global view of the LCR-manifold, we have to pass to the chart, which contains the entire LCR-manifold. That is, the bounded domain chart. The computations become complicated, but we are sure that we will not miss anything.

The plane $X^{11} = 0$ corresponds to the plane $Y^{11} + Y^{31} = 0$ in the bounded coordinates, the plane $X^{02} = 0$ corresponds to the plane $Y^{02} + Y^{22} = 0$, and the corresponding points in $G(4,2)$ are

$$Y^{ni} = \begin{pmatrix} Y^{01} & Y^{02} \\ Y^{11} & Y^{12} \\ Y^{21} & -Y^{02} \\ -Y^{11} & Y^{32} \end{pmatrix} := \begin{pmatrix} Y_1 \\ wY_1 \end{pmatrix} \tag{18.30}$$

in the bounded homogeneous coordinates Y^{nj}, where $w \in U(2)$ are the corresponding projective coordinates in the bounded realization of the "cartesian light-cone" LCR-manifold. We precisely have

$$\begin{pmatrix} Y^{21} & -Y^{02} \\ -Y^{11} & Y^{32} \end{pmatrix} = e^{i\tau} \begin{pmatrix} \cos\rho + i\sin\rho\cos\sigma & -i\sin\rho\sin\sigma\ e^{-i\chi} \\ -i\sin\rho\sin\sigma\ e^{i\chi} & \cos\rho - i\sin\rho\cos\sigma \end{pmatrix} \begin{pmatrix} Y^{01} & Y^{02} \\ Y^{11} & Y^{12} \end{pmatrix} \tag{18.31}$$

The two roots of the Kerr polynomial coincide for

$$\begin{array}{c} \det \begin{pmatrix} w_{11} + 1 & w_{12} \\ w_{21} & w_{22} + 1 \end{pmatrix} = 0 \\ \Downarrow \\ \cos\tau + \cos\rho = 0 \end{array} \tag{18.32}$$

which are the future and past celestial spheres, i.e. the (Penrose) scri+ and scri- respectively.

In the bounded realization we consider the following structure coordinates (normalization)

$$Y^{ni} = \begin{pmatrix} 1 & -w^{\widetilde{1}} \\ w^1 & 1 \\ w^0 & w^{\widetilde{1}} \\ -w^1 & w^{\widetilde{0}} \end{pmatrix} = \begin{pmatrix} 1 & \frac{iz^{\widetilde{1}}}{1-iz^{\widetilde{0}}} \\ \frac{iz^1}{1-iz^0} & 1 \\ \frac{1+iz^0}{1-iz^0} & \frac{-iz^{\widetilde{1}}}{1-iz^{\widetilde{0}}} \\ \frac{-iz^1}{1-iz^0} & \frac{1+iz^{\widetilde{0}}}{1-iz^{\widetilde{0}}} \end{pmatrix} \tag{18.33}$$

where their relation with the unbounded structure coordinates are written explicitly. The LCR-structure conditions are

$$1 - \overline{w^0}w^0 = 0 \quad , \quad \frac{w^{\widetilde{1}}}{1+w^{\widetilde{0}}} = \overline{\left(\frac{w^1}{1+w^0}\right)} \quad , \quad 1 - \overline{w^{\widetilde{0}}}w^{\widetilde{0}} = 0 \tag{18.34}$$

The ℓ^μ LCR-ray tracing is found by fixing s_1, s_2, s_3 and varying the affine parameter, taken to be σ, because

$$\begin{gathered} z^0 = \frac{\sin\tau - \sin\rho\cos\sigma}{\cos\tau + \cos\rho} = const \quad , \quad z^1 = \frac{\sin\rho}{\cos\tau\ + \cos\rho} \sin\sigma\ e^{i\chi} = const \\ s_1 := \frac{\sin\tau - \sin\rho\cos\sigma}{\cos\tau + \cos\rho} \quad , \quad s_2 := \frac{\sin\rho\sin\sigma}{\cos\tau\ + \cos\rho} \quad , \quad s_3 := \chi \\ z^{\widetilde{0}} = z^0 + 2s_2 \frac{\cos\sigma}{\sin\sigma} \end{gathered} \tag{18.35}$$

From

$$\begin{gathered} w_{11} = \frac{Y^{21}Y^{12} - Y^{11}Y^{22}}{Y^{01}Y^{12} - Y^{11}Y^{02}} \quad , \quad w_{12} = \frac{Y^{01}Y^{22} - Y^{21}Y^{02}}{Y^{01}Y^{12} - Y^{11}Y^{02}} \\ w_{21} = \frac{Y^{31}Y^{12} - Y^{11}Y^{32}}{Y^{01}Y^{12} - Y^{11}Y^{02}} \quad , \quad w_{22} = \frac{Y^{01}Y^{32} - Y^{31}Y^{02}}{Y^{01}Y^{12} - Y^{11}Y^{02}} \end{gathered} \tag{18.36}$$

we finally find

$$\begin{gathered} w_{11} = \frac{(1+s_1^2-s_2^2)\sin\sigma + 2s_2(s_1-i)\cos\sigma}{(1-2is_1-s_1^2+s_2^2)\sin\sigma - 2s_2(s_1+i)\cos\sigma} \quad , \quad w_{12} = \frac{-2is_2e^{-i\chi}\cos\sigma}{(1-2is_1-s_1^2+s_2^2)\sin\sigma - 2s_2(s_1+i)\cos\sigma} \\ w_{21} = \frac{-2is_2e^{i\chi}\cos\sigma}{(1-2is_1-s_1^2+s_2^2)\sin\sigma - 2s_2(s_1+i)\cos\sigma} \quad , \quad w_{22} = \frac{(1+s_1^2-s_2^2)\sin\sigma + 2s_2(s_1+i)\cos\sigma}{(1-2is_1-s_1^2+s_2^2)\sin\sigma - 2s_2(s_1+i)\cos\sigma} \end{gathered} \tag{18.37}$$

The n^μ LCR-ray tracing is found a completely analogous procedure.

18.3 An irreducible quadratic LCR-structure

Let us now turn to "flat" LCR-structure determined by the simple quadratic surface (in the unbounded Siegel realization)

$$K_U(X) = X^1X^2 - X^0X^3 = 0 \tag{18.38}$$

of $CP(3)$. It is the quadratic Kerr polynomial which is symmetric relative to z-rotations, time translations and dilations. This last scale invariance makes the present "flat" LCR-structure important. Apparently the quadric of $CP(3)$

is algebraically regular. But, we will see that its reduction to the real LCR-manifold is going to generated a non-permitting singularity. We have

$$\begin{array}{l} X^0 = 1 \quad , \quad X^1 = \lambda \quad , \quad X^2 = -i[(x^0 - x^3) - (x^1 - ix^2)\lambda] \\ X^3 = -i[-(x^1 + ix^2) + (x^0 + x^3)\lambda] \end{array} \tag{18.39}$$

The Kerr polynomial and its two solutions are

$$\begin{array}{l} (x^1 - ix^2)\lambda^2 + 2x^3\lambda - (x^1 + ix^2) = 0 \\ \lambda_{1,2} = \frac{-x^3 \pm \sqrt{\Delta}}{x - iy} \quad , \quad \Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 \end{array} \tag{18.40}$$

where $\lambda_{1,2}$ are the two values of λ on the two sheets of the quadric. The intersection of the two sheets of $CP(3)$ becomes

$$\Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 = 0 \tag{18.41}$$

in the LCR-submanifold of $G(4,2)$.

The preceding calculations are described as follows in the algebraic picture. The two points

$$\begin{array}{l} X^{n1} = \begin{pmatrix} 1 \\ \lambda_1(x) \\ -i[x^0 - x^3 - (x^1 - ix^2)\lambda_1] \\ -i[-(x^1 - ix^2) + (x^0 + x^3)\lambda_1] \end{pmatrix} \\ X^{n2} = \begin{pmatrix} 1 \\ \lambda_2(x) \\ -i[x^0 - x^3 - (x^1 - ix^2)\lambda_2] \\ -i[-(x^1 - ix^2) + (x^0 + x^3)\lambda_2] \end{pmatrix} \end{array} \tag{18.42}$$

of the above quadric belong to different sheets created by the considered projection and they correspond to a point x^a of the characteristic boundary $\mathbb{R}^4$ of the "upper half-plane" domain of $G(4,2)$. If $\det(\lambda^{Ai}) = \lambda_2 - \lambda_1 = 0$, the two points coincide, that is, the projection line is tangent to the quadric. Recall that in the general case with gravity, we would have $r^a(x) = x^a + iy^a(x)$ and the intersection would be

$$\begin{array}{l} (r^1)^2 + (r^2)^2 + (r^3)^2 = 0 \\ \\ (x^1)^2 + (x^2)^2 + (x^3)^2 - [(y^1)^2 + (y^2)^2 + (y^3)^2] = 0 \\ x^1y^1 + x^2y^2 + x^3y^3 = 0 \end{array} \tag{18.43}$$

generally a curve in $\mathbb{R}^4$. It is the LCR-submanifold corresponding to the complex 1-dimensional intersection curve of the two branches (:= the branch curve of the quadric) of $CP(3)$. The quadric is a well defined 2-dimensional complex surface of $CP(3)$ corresponding to the 1-dimensional Riemann surface in CP^2. Recall that the Riemann surface is constructed by making a branch cut with boundary the branch points (or a branch point and infinity) and glue the sheets properly. The corresponding construction of the present quadric in $CP(3)$ should be analogous. We take a branch cut of the surface (now) with boundary the branch curve (now) and glue the sheets properly. But this is not enough now, because this has to be reduced to the real LCR-manifold $\mathbb{R}^4$ in the Siegel chart (in the complete compact $U(2)$ spacetime in the Cartan chart will be described below). In the present case and with zero gravity ($y^a(x) = 0$) the branch curve is reduced to a point $\overrightarrow{x} = \overrightarrow{0}$. Therefore the branch cut should be reduced to a line joining $\overrightarrow{0}$ and ∞. The structure coordinates are

$$\begin{array}{lll} z^0 = iX^{21} = x^0 - |\overrightarrow{x}| & , & z^1 = \lambda_1 = \frac{|\overrightarrow{x}|-x^3}{x^1-ix^2} = \frac{x^1+ix^2}{|\overrightarrow{x}|+x^3} \\ z^{\widetilde{0}} = iX^{22} = x^0 + |\overrightarrow{x}| & , & z^{\widetilde{1}} = \frac{-1}{\lambda_2} = \frac{x^1-ix^2}{|\overrightarrow{x}|+x^3} = \overline{z^1} \end{array} \tag{18.44}$$

and the derived tetrad is

$$\begin{array}{l} \ell_\mu dx^\mu = \Lambda[|\overrightarrow{x}|dx^0 - \overrightarrow{x}\cdot d\overrightarrow{x}] \\ m_\mu dx^\mu = M[(|\overrightarrow{x}|(x^3+|\overrightarrow{x}|) - (x^1+ix^2)x^1)dx^1 + \\ \quad +(i|\overrightarrow{x}|(x^3+|\overrightarrow{x}|) - (x^1+ix^2)x^2)dx^2 - (x^1+ix^2)(x^3+|\overrightarrow{x}|)dx^3] \\ n_\mu dx^\mu = N[|\overrightarrow{x}|dx^0 + \overrightarrow{x}\cdot d\overrightarrow{x}] \end{array}$$

$$\ell\wedge m\wedge n\wedge \overline{m} = -4i|\overrightarrow{x}|^4(x^3+|\overrightarrow{x}|)^2 dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \neq 0, \quad \forall x^\mu \in \mathbb{R}^4 - \{\mathbb{R}_-\} \tag{18.45}$$

where the tetrad-Weyl factors are arbitrary as expected. The tetrad is singular (because it cannot be a basis of the tangent space) in the negative z-axis, where the branch cut in the algebraic quadric is reduced.

We can make the same calculations in the compact realization of complete spacetime. In this coordinate patch, Y^n is given by the linear transformation

$$\begin{array}{lll} X^0 = \frac{1}{\sqrt{2}}(Y^0+Y^2) & , & X^1 = \frac{1}{\sqrt{2}}(Y^1+Y^3) \\ X^2 = \frac{1}{\sqrt{2}}(Y^0-Y^2) & , & X^3 = \frac{1}{\sqrt{2}}(Y^1-Y^3) \end{array} \tag{18.46}$$

Then the Kerr polynomial has the same form

$$K_B(Y) = Y^1Y^2 - Y^0Y^3 \tag{18.47}$$

as in the unbounded realization. This quadratic LCR-struture has the same form in the bounded and unbounded realizations. In the bounded realization the homogeneous coordinates of $G(4,2)$ have the form

$$Y^{ni} = \begin{pmatrix} Y^{01} & Y^{02} \\ Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \\ Y^{31} & Y^{32} \end{pmatrix} = \begin{pmatrix} k \\ wk \end{pmatrix} \qquad \det k \neq 0 \quad , \quad w \in U(2) \tag{18.48}$$

where the 2x2 matrix w are the projective coordinates. Hence we will substitute

$$Y^n = \begin{pmatrix} 1 \\ k \\ w_{00} + w_{01}k \\ w_{10} + w_{11}k \end{pmatrix} \tag{18.49}$$

in the new (bounded) form of the Kerr quadric. Then it takes the form

$$k^2 w_{01} + k(w_{00} - w_{11}) - w_{10} = 0 \tag{18.50}$$

with singularities at the points

$$\begin{array}{c} w = e^{i\tau} I \quad and \quad w = -e^{i\tau} I \\ \Downarrow \\ \rho = 0 \quad and \quad \rho = \pi \end{array} \tag{18.51}$$

The ℓ^μ and n^μ rays, which pass from these two points are determined by $s_1 := \frac{\sin\tau \mp \sin\rho}{\cos\tau + \cos\rho} = \frac{\sin\tau}{\cos\tau\ \pm 1}$, $s_2 := \sigma$ and $s_3 := \chi$ respectively. Notice that the concentration of rays at the above points of $SU(2)$, creates an essential singularity at $|\overrightarrow{x}| = 0$, not permitting the smooth passage of the rays from the one $\mathbb{R}^4$-sheet to the other. Hence this LCR-manifold should be rejected, because it is not defined in the entire $U(2)$ universe. In the next section, we will see that the solitonic electron LCR-structure is well defined in PCFT, while it is **not** well defined in general relativity, because of its naked essential singularity!

18.4 Compatible metrics of the "flat" LCR-manifolds

Using the following spinorial form of the rank-2 matrix X^{mj} of the "flat" LCR-structure in its unbounded realization

$$X^{mj} = \begin{pmatrix} \lambda^{Aj} \\ -i x_{A'B} \lambda^{Bj} \end{pmatrix} \tag{18.52}$$

and the Kerr polynomial of the surface of $CP(3)$ we may compute and normalize two roots λ^{Aj}. Then the flat null tetrad

$$\begin{array}{l} L^a = \frac{1}{\sqrt{2}} \overline{\lambda}^{A'1} \lambda^{B1} \sigma^a_{A'B} \quad , \quad N^a = \frac{1}{\sqrt{2}} \overline{\lambda}^{A'2} \lambda^{B2} \sigma^a_{A'B} \quad , \quad M^a = \frac{1}{\sqrt{2}} \overline{\lambda}^{A'2} \lambda^{B1} \sigma^a_{A'B} \\ \det(\lambda^{Aj}) = \epsilon_{AB} \lambda^{A1} \lambda^{B2} = 1 \end{array} \tag{18.53}$$

determines the class of flat compatible metrics

$$[g_{\mu\nu}] = \Lambda N (L_\mu N_\nu + N_\mu L_\nu) - M\overline{M}(M_\mu \overline{M}_\nu + \overline{M}_\mu M_\nu) \tag{18.54}$$

Hence the two different LCR-structures, "natural $U(2)$" and "cartesian light-cone" are compatible with the two "flat" Cartan-Klein geometries based on $U(2)$ and Lorentz groups. It happens, because the flat $U(2)$ metric

$$\begin{array}{l} ds^2_B = (d\tau)^2 - (d\rho)^2 - \sin^2\rho (d\sigma)^2 - \sin^2\rho \sin^2\sigma (d\chi)^2 =: \widehat{\eta}_{\mu\nu} dx^\mu dx^\nu \\ \tau \in (-\pi, \pi) \quad , \quad \rho \in [0, 2\pi) \quad , \quad \sigma \in [0, \pi] \quad , \quad \chi \in [0, 2\pi) \end{array} \tag{18.55}$$

is conformally flat and connected relative to the affine Lorentz group. All its conformally equivalent metrics are regions of this spacetime as the Cayley coordinate transformation implies

$$\begin{aligned}
&\widehat{\eta}_{\mu\nu}dx^{\mu}dx^{\nu} := (\cos\tau - \cos\rho)^2\eta_{\mu\nu}dx^{\mu}dx^{\nu} \\
&(\cos\tau - \cos\rho)^2 = \tfrac{4}{1+2[(x^0)^2+|\overrightarrow{x}|^2]+[(x^0)^2-|\overrightarrow{x}|^2]^2}
\end{aligned} \tag{18.56}$$

De Sitter metric ds_S^2 is also conformally equivalent to the above metric

$$\begin{aligned}
&ds_S^2 = (dt)^2 - T_0^2\cosh^2\tfrac{t}{T_0}[(d\rho)^2 - \sin^2\rho(d\sigma)^2 - \sin^2\rho\sin^2\sigma(d\chi)^2] = \\
&\quad = T_0^2\cosh^2\tfrac{t}{T_0}[\widehat{\eta}_{\mu\nu}dx^{\mu}dx^{\nu}] \\
&\tau = 2\arctan(e^{\frac{t}{T_0}}) \quad , \quad T_0 := \sqrt{\tfrac{3}{\Lambda}}
\end{aligned} \tag{18.57}$$

but with $\rho \in [0, 2\pi)$. It covers the entire covering spacetime $R \times SU(2)$.

Hence the LCR-structure is not directly related to a precise lorentzian riemannian metric. It is better to imagine it as a pair of retarded and advanced "filaments" (ℓ^{μ}, n^{μ}). Such a filamentary structure is the characteristic property of the spacetimes based on the LCR-structure. The caustics of these integral curves will be related to the "matter".

19 THE ELECTRON LCR-MANIFOLD

Einstein's revolution was the consideration of a geometrical notion, the metric $g_{\mu\nu}$, as the fundamental quantity of nature. But when he wrote down his equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = T_{\mu\nu}$, he had to refer to the "matter" $T_{\mu\nu}$ as the second part. The fundamental quantity of pseudo-conformal field theory (PCFT) is more general notion than lorentzian Cauchy-Riemann (LCR) structure, which is essentially an integrability condition for a (global) basis $(\ell, m, n, \overline{m})$ of the tangent (and cotangent) space of the spacetime manifold. This generalization permitted us to write down a renormalizable metric independent action, which is not topological. But notice that only metrics, which admit two geodetic and shear-free null congruences exist in PCFT. The breaking of the fundamental tetrad-Weyl symmetry is imposed by the existence of a representative of the class, where the charge, energy-momentum, angular momentum conservations are valid. Hence we will say that the tetrad-Weyl symmetry is broken by the existence of the conservation laws.

The static electron is identified with the static axially symmetric LCR-structure determined with the linear trajectory $\xi^a = (\tau, 0, 0, ia)$. That is we have

$$X^{mi} = \begin{pmatrix} 1 & -z^{\widetilde{1}} \\ z^1 & 1 \\ -i(z^0 - ia) & i(z^{\widetilde{0}} - ia)z^{\widetilde{1}} \\ -i(z^0 + ia)z^1 & -i(z^{\widetilde{0}} + ia) \end{pmatrix} \tag{19.1}$$

where $(z^{\alpha}; z^{\widetilde{\beta}})$ are now the structure coordinates. Here I will first derive the "flat" LCR-structure (defined by $X^{\dagger}E_U X = 0$) and after I will make a "Kerr-Schild" ansatz adapted to the LCR-tetrad to finally refind the axially symmetric LCR-structure, which is identified with the electron. I think this approach will make general relativists more confident to the final picture of the electron as a gaussian beam (in the optics terminology) in $U(2)$ spacetime.

This procedure implies first the "flat" LCR-structure coordinates

$$\begin{array}{lcl} z^0 = t - r + ia\cos\theta & , & z^1 = e^{i\varphi}\tan\frac{\theta}{2} \\ z^{\widetilde{0}} = t + r - ia\cos\theta & , & z^{\widetilde{1}} = \frac{r+ia}{r-ia}e^{-i\varphi}\tan\frac{\theta}{2} \end{array} \tag{19.2}$$

from which we find the tetrad compatible with the Minkowski metric

$$\begin{array}{l} L_\mu dx^\mu = \Lambda[dt - dr - a\sin^2\theta d\varphi] \\ N_\mu dx^\mu = N[dt + \frac{r^2+2a^2\cos^2\theta - a^2}{r^2+a^2}dr - a\sin^2\theta\ d\varphi] \\ M_\mu dx^\mu = M[-ia\sin\theta\ (dt - dr) + (r^2 + a^2\cos^2\theta)d\theta + \\ \qquad +i\sin\theta(r^2+a^2)d\varphi] \end{array} \tag{19.3}$$

where the tetrad-Weyl factors are not determined, as expected. They are determined by simply imposing that the tetrad gives the Minkowski metric. But for that, we have to find first the relation of the cartesian coordinates with the present convenient coordinates (t, r, θ, φ), which we will call "asymmetric".

The general relation between the projective coordinates and the homogeneous coordinates of $G(4, 2)$ is found by simply inverting their definition formula. We finally find

$$\begin{array}{l} r^0 = i\frac{(X^{01}X^{32}-X^{31}X^{02})+(X^{21}X^{12}-X^{11}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})} \\ r^1 = i\frac{(X^{11}X^{32}-X^{31}X^{12})+(X^{21}X^{02}-X^{01}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})} \\ r^2 = \frac{(X^{11}X^{32}-X^{31}X^{12})-(X^{21}X^{02}-X^{01}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})} \\ r^3 = i\frac{(X^{01}X^{32}-X^{31}X^{02})-(X^{21}X^{12}-X^{11}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})} \end{array} \tag{19.4}$$

We already know that the imaginary part of $r^b = x^b + iy^b$ determines the gravitational "dressing", because the "flatness" condition implies $y^b = 0$. The Minkowski coordinates x^b are related with the "asymmetric" (t, r, θ, φ) via the relation

$$\begin{array}{l} x^0 = t \\ x^1 + ix^2 = (r - ia)\sin\theta e^{i\varphi} \\ x^3 = r\cos\theta \\ \\ r^4 - [(x^1)^2 + (x^2)^2 + (x^3)^2 - a^2]r^2 - a^2(x^3)^2 = 0 \\ \cos\theta = \frac{x^3}{r} \quad , \quad \sin\theta = \sqrt{\frac{(x^1)^2+(x^2)^2}{r^2+a^2}} \end{array} \tag{19.5}$$

with the following diffeomorphic relations

$$\begin{array}{l}
dx^0 = dt \\
dx^1 = \sin\theta\cos\varphi dr + \cos\theta(r\cos\varphi + a\sin\varphi)d\theta - \sin\theta(r\sin\varphi - a\cos\varphi)d\varphi \\
dx^2 = \sin\theta\sin\varphi dr + \cos\theta(r\sin\varphi - a\cos\varphi)d\theta + \sin\theta(r\cos\varphi + a\sin\varphi)d\varphi \\
dx^3 = \cos\theta dr - r\sin\theta d\theta
\end{array} \tag{19.6}$$

and

$$\begin{array}{l}
dt = dx^0 \\
dr = \frac{rx^1 - ax^2}{r^2+a^2}dx^1 + \frac{ax^1+rx^2}{r^2+a^2}dx^2 + \frac{x^3}{r}dx^3 \\
d\theta = \frac{x^3(rx^1-ax^2)}{r^2\sqrt{(r^2+a^2)((x^1)^2+(x^2)^2)}}dx^1 + \frac{x^3(ax^1+rx^2)}{r^2\sqrt{(r^2+a^2)((x^1)^2+(x^2)^2)}}dx^2 - \frac{\sqrt{(x^1)^2+(x^2)^2}}{r\sqrt{r^2+a^2}}dx^3 \\
d\varphi = -\frac{ax^1+rx^2}{r((x^1)^2+(x^2)^2)}dx^1 + \frac{rx^1-ax^2}{r((x^1)^2+(x^2)^2)}dx^2
\end{array} \tag{19.7}$$

Hence, we finally find that the conventional tetrad corresponding to the Minkowski metric

$$\begin{array}{l}
L_\mu dx^\mu = [dt - dr - a\sin^2\theta d\varphi] \\
N_\mu dx^\mu = \frac{r^2+a^2}{2(r^2+a^2\cos^2\theta)}[dt + \frac{r^2+2a^2\cos^2\theta - a^2}{r^2+a^2}dr - a\sin^2\theta\ d\varphi] \\
M_\mu dx^\mu = \frac{-1}{\sqrt{2}(r+ia\cos\theta)}[-ia\sin\theta\ (dt-dr) + (r^2+a^2\cos^2\theta)d\theta + \\
\qquad +i\sin\theta(r^2+a^2)d\varphi]
\end{array} \tag{19.8}$$

The general tetrad is found with the "Kerr-Schild" ansatz adapted to the LCR-structure formalism

$$\ell_\mu = L_\mu \quad , \quad m_\mu = M_\mu \quad , \quad n_\mu = N_\mu + \frac{h(r)}{2(r^2+a^2\cos^2\theta)}\ L_\mu \tag{19.9}$$

I want to point out that we find the same static LCR-structure looking for LCR-structures admitting time translation and axisymmetric symmetries.

With the above definition of the coordinates (t, r, θ, φ), the structure coordinates have the form

$$\begin{array}{l}
z^0 = t - r + ia\cos\theta \quad , \quad z^1 = e^{i\varphi}\tan\frac{\theta}{2} \\
z^{\widetilde{0}} = t + r - ia\cos\theta - 2f_1 \quad , \quad z^{\widetilde{1}} = \frac{r+ia}{r-ia}\ e^{2iaf_2}\ e^{-i\varphi}\tan\frac{\theta}{2}
\end{array} \tag{19.10}$$

where the two new functions are

$$f_1(r) = \int \frac{h}{r^2+a^2+h}\ dr \quad , \quad f_2(r) = \int \frac{h}{(r^2+a^2+h)(r^2+a^2)}\ dr \tag{19.11}$$

The Newman-Penrose spin coefficients are found to be

$$\begin{array}{|l|}
\hline
\alpha = \frac{ia(1+\sin^2\theta) - r\cos\theta}{2\sqrt{2}\sin\theta\ (r-ia\cos\theta)^2} \quad , \quad \beta = \frac{\cos\theta}{2\sqrt{2}\sin\theta\ (r+ia\cos\theta)} \\
\hline
\gamma = -\frac{a^2+iar\cos\theta+h}{2\rho^2\ (r-ia\cos\theta)} + \frac{h'}{4\rho^2} \quad , \quad \varepsilon = 0 \\
\hline
\mu = -\frac{r^2+a^2+h}{2\rho^2\ (r-ia\cos\theta)} \quad , \quad \pi = \frac{ia\sin\theta}{\sqrt{2}(r-ia\cos\theta)^2} \\
\hline
\rho = -\frac{1}{r-ia\cos\theta} \quad , \quad \tau = -\frac{ia\sin\theta}{\sqrt{2}\rho^2} \\
\hline
\kappa = 0 \quad , \quad \sigma = 0 \quad , \quad \nu = 0 \quad , \quad \lambda = 0 \\
\hline
\end{array} \tag{19.12}$$

which will be useful for our computations. Recall that the Kerr-Newman spacetime has $h(r) = -2Mr + e^2$. In this case the integrals are

$$\begin{array}{l}
f_1(r) = \int \frac{-2Mr+e^2}{r^2+a^2-2Mr+e^2}\, dr = -M \ln \frac{|\Delta|}{r1} + \frac{2M^2-e^2}{\Theta} \arctan \frac{\Theta}{r-M} \\
f_2(r) = \int \frac{-2Mr+e^2}{(r^2+a^2-2Mr+e^2)(r^2+a^2)}\, dr = \frac{1}{2ia} \ln[r_2 \frac{r-ia}{r+ia} (\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}] \\
\\
\Delta := r^2 + a^2 - 2Mr + e^2 \quad , \quad \Theta := \sqrt{a^2 + e^2 - M^2}
\end{array} \tag{19.13}$$

and the structure coordinates of the "Kerr-Newman" LCR-manifold are

$$\begin{array}{l}
z^0 = t - r + ia\cos\theta \quad , \quad z^1 = e^{i\varphi} \tan\frac{\theta}{2} \\
z^{\widetilde{0}} = t + r - ia\cos\theta + 2M \ln \frac{|\Delta|}{r_1} + \frac{2(e^2-2M^2)}{\Theta} \arctan \frac{\Theta}{r-M} \\
z^{\widetilde{1}} = r_2 (\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}\ e^{-i\varphi} \tan\frac{\theta}{2}
\end{array} \tag{19.14}$$

in the Lindquist coordinates (t, r, θ, φ). The constants r_1 and r_2 are normalization constants. Notice the singularities in the ambient complex manifold occur at the two complex values of $r = M \pm i\Theta$. It is well known to general relativists that this choice of tetrad-Weyl factors preserve the electromagnetic current and the energy-momentum and angular momentum currents. This breaking of the tetrad-Weyl symmetry will be shown in the subsections of the electromagnetic and gravitational "dressings" of the Kerr-Newman manifold.

The general form (4.13)

$$\begin{array}{l}
\rho_{11}(\overline{z^\alpha}, z^\beta) = 0 \quad , \quad \rho_{12}(\overline{z^\alpha}, z^{\widetilde{\beta}}) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\beta}}) = 0 \\
\\
\frac{\partial \rho_{ij}}{\partial z^b} \neq 0 \neq \frac{\partial \rho_{ij}}{\partial \overline{z^b}}
\end{array} \tag{19.15}$$

of the embedding of the LCR-manifold in the ambient complex manifold may be viewed as a deformation of the 3-dimensional CR-manifold $\rho_{11}(\overline{z^\alpha}, z^\beta) = 0$ through a formal **anti-**meromorphic transformation

$$z^{\widetilde{\beta}} = f^{\widetilde{\beta}}(\overline{z^\alpha}; s) \tag{19.16}$$

which generalizes the trivial transformation of the degenerate LCR-structure. In the present electron LCR-structure this deformation takes the form

$$\begin{array}{l}
z^{\widetilde{0}} = \overline{z^0} + 2(r - f_1) \\
z^{\widetilde{1}} = r_2 \overline{z^1} (\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}
\end{array} \tag{19.17}$$

where the deformation parameter is the real variable r.

The static axially symmetric LCR-structure (identified with the electron) is stable, because all its relative invariants

$$\begin{array}{l}
\Phi_1 = \frac{\rho-\overline{\rho}}{i} = \frac{-2a\cos\theta}{r^2+a^2\cos^2\theta} \\
\Phi_2 = \frac{\mu-\overline{\mu}}{i} = -\frac{(r^2+a^2+h)a\cos\theta}{(r^2+a^2\cos^2\theta)^2} \\
\Phi_3 = -(\tau + \overline{\pi}) = \frac{2iar\sin\theta}{\sqrt{2}(r+ia\cos\theta)^2(r-ia\cos\theta)}
\end{array} \tag{19.18}$$

do not vanish. Notice the gravitational and electromagnetic "dressings" that this soliton contains.

Making all the preceding calculations and comments, I had in my mind the values of the electron parameters. Therefore let us make them precise, and compute[5] the electron parameters in the dimensionless units (DU) $c = G = \hbar = 1$. These constants in the SI system have the values

$$\begin{array}{l} c = 2.99792458 * 10^8 \ m * s^{-1} \\ G = 6.674 * 10^{-11} m^3 * kg^{-1} * s^{-2} \\ \hbar = \frac{h}{2\pi} = 1.05 * 10^{-34} kg * m^2 * s^{-1} \\ M_P = \sqrt{\frac{\hbar c}{G}} = 2.176 * 10^{-8} kg \end{array} \tag{19.19}$$

Then the electron mass M_e, charge e^2 and spin parameter a have the values

$$\begin{array}{l} M = \frac{M_e}{M_P} = 4.18 * 10^{-23} \\ e^2 = \frac{q^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137} \\ a = \frac{\hbar}{2M_e} = 2.09 * 10^{23} \\ \\ a^2 >> e^2 >> M^2 \end{array} \tag{19.20}$$

Hence $a^2 + e^2 - M^2 > 0$, which justifies my calculational choices.

We see that the electron metric has an essential naked singularity. This is a problem for general relativity, because its fundamental quantity, the metric, does not "see" the algebraic structure. It is known (and well described in many books of general relativity) that its analytic extension has two sheets x^b and x'^b which are determined by the two roots

$$r = \pm \left\{ \frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2} + \sqrt{[\frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2}]^2 + a^2(x^3)^2} \right\}^{\frac{1}{2}} \tag{19.21}$$

In the next subsections, I will show that these two surfaces constitute the boundary $U(2)$ of the bounded realization of the $SU(2,2)$ classical domain and their correspondence is the well known Cayley transformation. It will make clear why the spinorial electron naked singularity in $U(2)$ universe can be properly incorporated in PCFT, while it is rejected as "unphysical" by the riemannian formalism.

19.1 The tetrad-Weyl connection for the electron LCR-structure

In the general case of a LCR-tetrad

$$\begin{array}{l} d\ell - Z_1 \wedge \ell = i\Phi_1 m \wedge \overline{m} \\ dn - Z_2 \wedge n = i\Phi_2 m \wedge \overline{m} \\ dm - Z_3 \wedge m = \Phi_3 \ell \wedge n \end{array} \tag{19.22}$$

where the abelian connection of the tetrad-Weyl transformation is

$$
\begin{aligned}
Z_{1\mu} = & (\theta_1 + \mu + \overline{\mu})\ell_\mu + (\varepsilon + \overline{\varepsilon})n_\mu - (\alpha + \overline{\beta} - \overline{\tau})m_\mu - \\
& -(\overline{\alpha} + \beta - \tau)\overline{m}_\mu \\
Z_{2\mu} = & -(\gamma + \overline{\gamma})\ell_\mu + (\theta_2 - \rho - \overline{\rho})n_\mu - (\pi - \alpha - \overline{\beta})m_\mu - \\
& -(\overline{\pi} - \overline{\alpha} - \beta)\overline{m}_\mu \\
Z_{3\mu} = & (\gamma - \overline{\gamma} + \overline{\mu})\ell_\mu + (\varepsilon - \overline{\varepsilon} - \rho)n_\mu - (\theta_3 + \pi - \overline{\tau})m_\mu - \\
& -(\beta - \overline{\alpha})\overline{m}_\mu
\end{aligned}
\tag{19.23}
$$

with the functions θ_1 , θ_2 , θ_3 á priori arbitrary, and the torsion is

$$
\Phi_1 = \tfrac{\rho - \overline{\rho}}{i} \quad , \quad \Phi_2 = \tfrac{\mu - \overline{\mu}}{i} \quad , \quad \Phi_3 = -(\tau + \overline{\pi})
\tag{19.24}
$$

which get precise transformations under the tetrad-Weyl transformations. The spin coefficients transform as follows

$$
\begin{aligned}
\alpha' = & \tfrac{1}{M}\alpha + \tfrac{M\ \overline{M} - \Lambda N}{4M\Lambda N}(\overline{\tau} + \pi) + \tfrac{1}{4M}\overline{\delta}\ln\tfrac{\Lambda}{N\overline{M}^2} \\
\beta' = & \tfrac{1}{\overline{M}}\beta + \tfrac{M\ \overline{M} - \Lambda N}{4\overline{M}\Lambda N}(\tau + \overline{\pi}) + \tfrac{1}{4\overline{M}}\delta\ln\tfrac{\Lambda M^2}{N} \\
\gamma' = & \tfrac{1}{\Lambda}\gamma + \tfrac{M\ \overline{M} - \Lambda N}{4M\ \overline{M}\Lambda}(\overline{\mu} - \mu) + \tfrac{1}{4\Lambda}\Delta\ln\tfrac{M}{N^2\overline{M}} \\
\varepsilon' = & \tfrac{1}{N}\varepsilon + \tfrac{M\ \overline{M} - \Lambda N}{4M\ \overline{M}N}(\overline{\rho} - \rho) + \tfrac{1}{4N}D\ln\tfrac{M\Lambda^2}{\overline{M}} \\
\mu' = & \tfrac{1}{2\Lambda}(\mu + \overline{\mu}) + \tfrac{N}{2M\ \overline{M}}(\mu - \overline{\mu}) + \tfrac{1}{2\Lambda}\Delta\ln(M\ \overline{M}) \\
\rho' = & \tfrac{1}{2N}(\rho + \overline{\rho}) + \tfrac{\Lambda}{2M\ \overline{M}}(\rho - \overline{\rho}) - \tfrac{1}{2N}D\ln(M\ \overline{M}) \\
\pi' = & \tfrac{\overline{M}}{2\Lambda N}(\pi + \overline{\tau}) + \tfrac{1}{2M}(\pi - \overline{\tau}) + \tfrac{1}{2M}\overline{\delta}\ln(\Lambda N) \\
\tau' = & \tfrac{M}{2\Lambda N}(\tau + \overline{\pi}) + \tfrac{1}{2\overline{M}}(\tau - \overline{\pi}) - \tfrac{1}{2\overline{M}}\delta\ln(\Lambda N) \\
\kappa' = & \tfrac{\Lambda}{N\overline{M}}\kappa \quad , \quad \sigma' = \tfrac{M}{N\overline{M}}\sigma \\
\nu' = & \tfrac{N}{\Lambda M}\nu \quad , \quad \lambda' = \tfrac{\overline{M}}{\Lambda M}\lambda
\end{aligned}
\tag{19.25}
$$

From this point of view the connection $Z_{i\mu}$ transforms as

$$
Z'_{1\mu} = Z_{1\mu} + \partial_\mu \ln\Lambda \quad , \quad Z'_{2\mu} = Z_{2\mu} + \partial_\mu \ln N \quad , \quad Z'_{3\mu} = Z_\mu + \partial_\mu \ln M
$$

$$
\begin{aligned}
& \theta'_1 = \tfrac{1}{\Lambda}\theta_1 + \tfrac{1}{\Lambda}n^\mu\partial_\mu \ln\tfrac{\Lambda}{M\overline{M}} \quad , \quad \theta'_2 = \tfrac{1}{N}\theta_2 + \tfrac{1}{N}\ell^\mu\partial_\mu \ln\tfrac{N}{M\overline{M}} \\
& \theta'_3 = \tfrac{1}{M}\theta_3 + \tfrac{1}{M}\overline{m}^\mu\partial_\mu \ln\tfrac{M}{\Lambda N}
\end{aligned}
\tag{19.26}
$$

and the torsion Φ_1, Φ_2, Φ_3 as

$$
\begin{aligned}
\rho' - \overline{\rho'} = & \tfrac{\Lambda}{M\overline{M}}(\rho - \overline{\rho}) \\
\mu' - \overline{\mu'} = & \tfrac{N}{M\overline{M}}(\mu - \overline{\mu}) \\
\tau' + \overline{\pi'} = & \tfrac{M}{\Lambda N}(\tau + \overline{\pi})
\end{aligned}
\tag{19.27}
$$

Hence the differential forms

$$
F_1 = dZ_1 \quad , \quad F_2 = dZ_2 \quad , \quad F = dZ_3
\tag{19.28}
$$

are LCR invariants. In the generic case of non vanishing relative invariants, the gauge transformations are satisfied if

$$
\theta_1 = n^\mu\partial_\mu \ln\Phi_1 \quad , \quad \theta_2 = \ell^\mu\partial_\mu \ln\Phi_2 \quad , \quad \theta_3 = \overline{m}^\mu\partial_\mu \ln\Phi_3
\tag{19.29}
$$

I point out that the real quantity

$$i(\rho-\overline{\rho})(\mu-\overline{\mu})(\tau+\overline{\pi})(\overline{\tau}+\pi)\ell\wedge m\wedge n\wedge\overline{m} \tag{19.30}$$

is a LCR structure invariant.

19.2 Integral curves of the flatprint LCR-structure

We have already found that in the flatprint electron LCR-structure the structure coordinates are

$$\begin{array}{lcl} z^0 = t-r+ia\cos\theta & , & z^1 = e^{i\varphi}\tan\frac{\theta}{2} \\ z^{\widetilde{0}} = t+r-ia\cos\theta & , & z^{\widetilde{1}} = \frac{r+ia}{r-ia}e^{-i\varphi}\tan\frac{\theta}{2} \end{array} \tag{19.31}$$

and the cartesian coordinates are

$$\begin{array}{l} x^0 = t \\ x^1+ix^2 = (r-ia)\sin\theta e^{i\varphi} \\ x^3 = r\cos\theta \\ \\ r^4-[(x^1)^2+(x^2)^2+(x^3)^2-a^2]r^2-a^2(x^3)^2=0 \end{array} \tag{19.32}$$

Then the $L^\mu\partial_\mu z^\alpha=0$ implies that the outgoing integral curves (rays) are determined by the surfaces

$$s_1 := t-r \quad , \quad s_2 := \theta \quad , \quad s_3 := \varphi \tag{19.33}$$

Assuming t as the parameter along the ray we find the congruence

$$\begin{array}{l} x_L^0(t) = t \\ x_L^1(t) = [(t-s_1)\cos\varphi + a\sin\varphi]\sin\theta \\ x_L^2(t) = [(t-s_1)\sin\varphi - a\cos\varphi]\sin\theta \\ x_L^3(t) = (t-s_1)\cos\theta \\ \\ Jacobian = [(t-s_1)^2+a^2\cos^2\theta]\sin\theta \end{array} \tag{19.34}$$

The variables (t,s_1,s_2,s_3) are the (natural) ray coordinates. The caustic is the surface where these coordinates fail to describe uniquely the rays. That is, where the jacobian of the coordinates (relative to the cartesian coordinates) vanishes. It occurs at

$$\begin{array}{l} \{t-s_1=0, \quad \theta=\frac{\pi}{2}\}\cup\{\theta=0\} \\ i.e. \\ \{(x^1)^2+(x^2)^2=a^2, \quad x^3=0\}\cup\{x^1=0=x^2\} \end{array} \tag{19.35}$$

which is written in cartesian coordinates. At $t=0$, the rays determined by $(0,s_1,s_2=\theta,s_3=\varphi)$ will cross the $t=0$ surface at

$$\begin{array}{l} x_L^0(0) = 0 \\ x_L^1(0) = (-s_1\cos\varphi + a\sin\varphi)\sin\theta \\ x_L^2(0) = (-s_1\sin\varphi - a\cos\varphi)\sin\theta \\ x_L^3(0) = -s_1\cos\theta \end{array} \tag{19.36}$$

Notice that the parameter t does not determine the distance of the points of a ray from the caustic. That is (t, s_1, s_2, s_3) are not caustic coordinates. Therefore it is convenient to use the caustic coordinates (r, s_1, s_2, s_3), which have the property $(0, s_1, \frac{\pi}{2}, s_3)$ to be on the caustic. In this caustic coordinate system the LCR-rays are traced by the relation

$$\begin{array}{l}
x_L^0(r) = s_1 + r \\
x_L^1(r) = (r\cos\varphi + a\sin\varphi)\sin\theta \\
x_L^2(r) = (r\sin\varphi - a\cos\varphi)\sin\theta \\
x_L^3(r) = r\cos\theta \\
\\
Jacobian = [r^2 + a^2\cos^2\theta]\sin\theta
\end{array} \tag{19.37}$$

The source of the LCR-rays are at $r = 0$, i.e.

$$\begin{array}{l}
x_L^0(0) = s_1 \\
x_L^1(0) = a\sin\varphi\sin\theta \\
x_L^2(0) = -a\cos\varphi\sin\theta \\
x_L^3(0) = 0
\end{array} \tag{19.38}$$

the disk found above.

The $N^\mu\partial_\mu z^{\widetilde{\alpha}} = 0$ implies that its incoming rays are determined by the surfaces

$$s_1' := t + r \quad , \quad s_2' := \theta \quad , \quad s_3' := \varphi + \arctan\tfrac{2ar}{a^2 - r^2} \tag{19.39}$$

Then we find the congruence

$$\begin{array}{l}
x_N^0(r) = s_1' - r \\
x_N^1(r) = [r\cos s_3' - a\sin s_3']\sin\theta \\
x_N^2(r) = [r\sin s_3' + a\cos s_3']\sin\theta \\
x_N^3(r) = r\cos\theta \\
\\
Jacobian = [r^2 + a^2\cos^2\theta]\sin\theta
\end{array} \tag{19.40}$$

As expected the velocities $\dot{x}_L^i(t)$ and $\dot{x}_N^i(t)$ have asymptotically opposite radial directions.

We will now show that the origin of the essential singularity of the Kerr manifold is the intersection of the two sheets of the static electron regular quadric (in the unbounded Siegel realization)

$$X^1X^2 - X^0X^3 + 2aX^0X^1 = 0 \tag{19.41}$$

of $CP(3)$. In the flatprint case we have

$$\begin{array}{l}
X^0 = 1 \quad , \quad X^1 = \lambda \quad , \quad X^2 = -i[(x^0 - x^3) - (x^1 - ix^2)\lambda] \\
X^3 = -i[-(x^1 + ix^2) + (x^0 + x^3)\lambda]
\end{array} \tag{19.42}$$

and the Kerr polynomial and its two solutions are

$$\begin{array}{l} (x^1 - ix^2)\lambda^2 + 2(x^3 - ia)\lambda - (x^1 + ix^2) = 0 \\ \lambda_{1,2} = \frac{-(x^3 - ia) \pm \sqrt{\Delta}}{x - iy} \quad , \quad \Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 - a^2 - 2iax^3 \end{array} \tag{19.43}$$

where $\lambda_{1,2}$ are the two values of λ on the two sheets of the quadric. The intersection curve of these two sheets is

$$\begin{array}{l} \Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 - a^2 - 2iax^3 = 0 \\ \\ x^3 = 0 \quad , \quad (x^1)^2 + (x^2)^2 = a^2 \end{array} \tag{19.44}$$

which, after the LCR projection to $\mathbb{R}^4$, becomes the singularity ring of the electron (Kerr-Newman) manifold. Notice that the quadratic surface is regular and the intersection of the two branches is implied by the projection. The points of the algebraic intersection curve (the branch curve) of the (regular) quadric of $CP(3)$ are regular points like any other point of the quadric.

19.3 Electron LCR-ray tracing

In the context of PCFT, the electron is a static and axially symmetric LCR-structure $(\ell, m; n, \overline{m})$ determined by a quadratic surface of $CP(3)$. The spacetime (universe) of the electron is $U(2)$, the double cover of $\mathbb{R}^4$ which communicate through conformal infinity and the unit disk $x^2 + y^2 < a^2, \quad z = 0$, located on the boundary $r = 0$. The rays x^μ_ℓ of the retarded congruence ℓ^μ are labeled by (u, θ, φ) defined by

$$z^0 =: u + iU(\theta, \varphi) \quad , \quad z^1 =: e^{i\varphi} \tan \tfrac{\theta}{2} \tag{19.45}$$

and the rays x^μ_n of the advanced congruence n^μ are labeled by (v, θ', φ') defined by

$$z^{\widetilde{0}} =: v + iV(\theta', \varphi') \quad , \quad z^{\widetilde{1}} =: e^{i\varphi'} \tan \tfrac{\theta'}{2} \tag{19.46}$$

If we define "natural time" $t := \frac{u+v}{2}$ and "natural distance" $r := \frac{v-u}{2}$, the retarded rays emerge from $r < 0$ through the unit disk of the electron universe $U(2)$, they reach conformal infinity through which pass again in $r < 0$ region and they come down (as advanced rays) to the unit disk, from which they pass to the $r > 0$ region and close their loop trace. The advanced rays emerge from conformal infinity, they come down and pass through the unit disk in the $r < 0$ region where they travel (as retarded rays) to conformal infinity and close their trace. That is the electron is like a gaussian beam of advanced and retarded rays in $U(2)$ with its neck at the unit disk on the boundary $r = 0$.

The two fundamental equations of the LCR-structure take the conserved ray-density forms

$$\begin{array}{c} dz^0 \wedge d\overline{z^0} \wedge dz^1 \wedge d\overline{z^1} = 0 \quad , \quad dz^{\widetilde{0}} \wedge d\overline{z^{\widetilde{0}}} \wedge dz^{\widetilde{1}} \wedge d\overline{z^{\widetilde{1}}} = 0 \\ \Downarrow \\ d(iU du \wedge dz^1 \wedge d\overline{z^1}) = 0 \quad , \quad d(iV dv \wedge dz^{\widetilde{1}} \wedge d\overline{z^{\widetilde{1}}}) = 0 \end{array} \tag{19.47}$$

If the LCR-structure has definite chirality $U=0$ or $V=0$, its ℓ^{μ} and n^{μ} LCR-rays density vanishes respectively. In the case of the electron and because of its left-right symmetry $z^1\overline{z^1}=\widetilde{z^1}\overline{\widetilde{z^1}}$ both retarded and advanced LCR-rays have the same densities found to be

$$
\begin{array}{c}
d[-2ia\frac{z^1\overline{z^1}}{1+z^1\overline{z^1}}d(t-r)\wedge dz^1\wedge d\overline{z^1}]=0 \quad , \quad d[2ia\frac{z^1\overline{z^1}}{1+z^1\overline{z^1}}d(t+r)\wedge d\widetilde{z^1}\wedge d\overline{\widetilde{z^1}}]=0 \\
\Downarrow \\
\rho(r,\theta,\varphi)\Delta S=\frac{a\sin^2\frac{\theta}{2}}{r\cos^4\frac{\theta}{2}}(r^2\sin\theta d\theta d\varphi) \quad , \quad \rho'(r,\theta',\varphi')\Delta S=\frac{a\sin^2\frac{\theta'}{2}}{r\cos^4\frac{\theta'}{2}}(r^2\sin\theta' d\theta' d\varphi')
\end{array}
\tag{19.48}
$$

but as we have proved above the ℓ^{μ} and n^{μ} LCR-rays have opposite directions through the singular and different tubes (the Penrose scri$\pm$ at the "infinity" of each $\mathbb{R}^4$-sheet. Also notice that the flow of the density decreases as $\frac{1}{r}$.

In the natural $(u,\ r,\ \theta,\ \varphi)$ coordinates of the ℓ^{μ} congruence the tetrad takes the form

$$
\begin{array}{l}
\ell_{\mu}dx^{\mu}=du-a\sin^2\theta\ d\varphi \\
n_{\mu}dx^{\mu}=\frac{1}{2(r^2+a^2\cos^2\theta)}[\Delta du+2(r^2+a^2\cos^2\theta)dr-a\Delta\sin^2\theta\ d\varphi] \\
m_{\mu}dx^{\mu}=\frac{-1}{\sqrt{2}(r+ia\cos\theta)}[-ia\sin\theta\ du+(r^2+a^2\cos^2\theta)d\theta+ \\
\qquad +i\sin\theta(r^2+a^2)d\varphi] \\
\\
\Delta:=r^2+a^2-2Mr+e^2
\end{array}
\tag{19.49}
$$

In the singularity disk the tetrad is not linearly independent. The disk singularity is a manifestation of the break down of the linear independence of the LCR-tetrad, which is a consequence of the coincidence of the two intersection points of the quadric surface of $CP(3)$ with its projection line. In the context of algebraic geometry it is a coordinate singularity related to the chosen affine space.

Already in the context of general relativity it has been observed that the mass does not permit an identification between the Penrose scri+ and scri- boundaries of spacetime. In order to make things explicit the Kerr-Newman integrable null tetrad will be used as an example. Around scri+ the coordinates $(u,\ w=\frac{1}{r},\ \theta,\ \varphi)$ are used, where the integrable tetrad takes the form

$$
\begin{array}{l}
\ell=du-a\sin^2\theta\ d\varphi \\
\\
n=\frac{1-2mw+e^2w^2+a^2w^2}{2w^2(1+a^2w^2\cos^2\theta)}[w^2\ du-\frac{2(1+a^2w^2\cos^2\theta)}{1-2mw+e^2w^2+a^2w^2}\ dw-aw^2\sin^2\theta\ d\varphi] \\
\\
m=\frac{1}{\sqrt{2}w(1+iaw\cos\theta)}[iaw^2\sin\theta\ du-(1+a^2w^2\cos^2\theta)\ d\theta- \\
\qquad -i\sin\theta(1+aw^2)\ d\varphi]
\end{array}
\tag{19.50}
$$

The physical space is for $w>0$ and the integrable tetrad is regular on scri+ up to a factor, which does not affect the congruence, and it can be regularly extended to $w<0$. Around scri- the coordinates $(v,\ w',\ \theta',\ \varphi')$ are used with

$$
\begin{array}{l}
dv = du + \frac{2(r^2+a^2)}{r^2-2mr+e^2+a^2}\ dr \\
dw' = -dw \quad , \quad d\theta' = d\theta \\
d\varphi' = d\varphi + \frac{2a}{r^2-2mr+e^2+a^2}\ dr
\end{array}
\tag{19.51}
$$

and the integrable tetrad takes the form

$$
\begin{array}{l}
\ell = \frac{1}{w'^2}[w'^2\ dv - \frac{2(1+a^2w'^2\cos^2\theta)}{1+2mw'+e^2w'^2+a^2w'^2}\ dw' - aw'^2\sin^2\theta'\ d\varphi'] \\
n = \frac{1+2mw'+e^2w'^2+a^2w'^2}{2(1+a^2w'^2\cos^2\theta')}[dv - a\sin^2\theta'\ d\varphi'] \\
m = \frac{-1}{\sqrt{2}w'(1-iaw'\cos\theta')}[iaw'^2\sin\theta\ dv - (1+a^2w'^2\cos^2\theta')\ d\theta' - \\
\quad -i\sin\theta'(1+aw'^2)\ d\varphi']
\end{array}
\tag{19.52}
$$

The physical space is for $w < 0$ and the integrable tetrad is regular on scri- up to a factor, which does not affect the congruence, and it can be regularly extended to $w > 0$. When $m \neq 0$ these two regions cannot be identified and the LCR-structure is extended across scri+ and scri- of the two $\mathbb{R}^4$-sheets . If the mass term vanishes the two regions of scri+ and scri- can be identified and the ℓ^μ and n^μ congruences are interchanged, when scri+ (and scri- respectively) is crossed, but the implied flat LCR-manifold will be singular, because of the remaining disk singularity.

19.4 Bounded realization of the flatprint LCR-structure

In the Lindquist coordinates, the ring naked singularity $r^2 + a^2\cos^2\theta = 0$ does not depend on the gravitational parameters of the electron. This is purely a spinorial effect, already present in the "flatprint" of the electron LCR-structure. Therefore, we will neglect gravitational and electromagnetic "dressings" and we will work with the "flatprint" of the electron LCR-structure. Recall that the one $\mathbb{R}^4$ sheet of the $U(2) \to \mathbb{R}^4$ Cayley $2 \to 1$ transformation is

$$
\begin{array}{l}
For\ s := R_0\frac{\sin\rho}{\cos\tau+\cos\rho} > 0 \\
x^0 = T_0\frac{\sin\tau}{\cos\tau+\cos\rho} \\
x^1 + ix^2 = R_0\frac{\sin\rho}{\cos\tau+\cos\rho}\sin\sigma\ e^{i\chi} \\
x^3 = R_0\frac{\sin\rho}{\cos\tau+\cos\rho}\cos\sigma
\end{array}
\tag{19.53}
$$

and the second $\mathbb{R}^4$ is identified with $s < 0$,

$$
\begin{array}{l}
For\ s := R_0\frac{\sin\rho}{\cos\tau+\cos\rho} < 0 \\
x'^0 = T_0\frac{\sin\tau}{\cos\tau+\cos\rho} \\
x'^1 + ix'^2 = -R_0\frac{\sin\rho}{\cos\tau+\cos\rho}\sin\sigma\ e^{i\chi} \\
x'^3 = -R_0\frac{\sin\rho}{\cos\tau+\cos\rho}\cos\sigma
\end{array}
\tag{19.54}
$$

The constants T_0 and R_0 are related to the time and space sizes. Notice that this is the Penrose artificial compactification of the Minkowski spacetime, but in the context of PCFT, this is implied by the formalism itself. In the case of the Penrose artificial compactification these two sheets $s \gtrless 0$ communicate through the scri+ and scri- infinities. In the case of the electron flatprint LCR-structure, these two sheets communicate through the glued two discs $(x^1)^2 + (x^2)^2 < a^2$ too, because we may assume

$$\begin{array}{l} r = +\left\{\frac{s^2-a^2}{2} + \sqrt{[\frac{s^2-a^2}{2}]^2 + a^2(x^3)^2}\right\}^{\frac{1}{2}} \quad for\ s > 0 \\ r = -\left\{\frac{s^2-a^2}{2} + \sqrt{[\frac{s^2-a^2}{2}]^2 + a^2(x^3)^2}\right\}^{\frac{1}{2}} \quad for\ s < 0 \end{array} \tag{19.55}$$

Notice that in the identified region (the disc for both sheets) $r = 0$ in both sheets. That is, $r = 0$ occurs at $x^3 = 0$ and $s^2 \leq a^2$ for both sheets $s \gtrless 0$.

The two LCR-congruences $\ell^\mu = \frac{dx_\ell^\mu}{dr}$ and $n^\mu = \frac{dx_n^\mu}{dr}$ of the flatprint electron LCR-manifold can be easily implied from the calculations of the previous section. The starting idea is that the structure coordinates $z^\alpha(x)$ provide the three invariants (s_1, s_2, s_3) along the ray, which label the ℓ-ray $x_\ell^\mu(r)$, and the structure coordinates $z^{\widetilde{\alpha}}(x)$ provide the invariants (s'_1, s'_2, s'_3), which label the n-ray $x_n^\mu(r)$. Hence we simply have the same forms, but we let $r \in (-\infty, +\infty)$ and at $r = 0$ we pass to the second $x_L'^\mu(r), x_L'^\mu(r) \in \mathbb{R}^4$ sheet. It is intuitively instructive.

The second way is tracing the rays $w_{L,N}(r; s_1, s_2, s_3) \in U(2)$ in the complete bounded universe $U(2)$ taking $r \in (-\infty, +\infty)$ as the parameter indicating the ray points.

From the relation

$$\begin{array}{lll} Y^0 = \frac{1}{\sqrt{2}}(X^0 + X^2) & , & Y^1 = \frac{1}{\sqrt{2}}(X^1 + X^3) \\ Y^2 = \frac{1}{\sqrt{2}}(X^0 - X^2) & , & Y^3 = \frac{1}{\sqrt{2}}(X^1 - X^3) \end{array} \tag{19.56}$$

between the bounded Y^{ni} and unbounded X^{ni} homogeneous coordinates and

$$X^{mi} = \begin{pmatrix} 1 & -z^{\widetilde{1}} \\ z^1 & 1 \\ -i(z^0 - ia) & i(z^{\widetilde{0}} - ia)z^{\widetilde{1}} \\ -i(z^0 + ia)z^1 & -i(z^{\widetilde{0}} + ia) \end{pmatrix} \tag{19.57}$$

we find

$$Y^{mi} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - i(z^0 - ia) & (-1 + i(z^{\widetilde{0}} - ia))z^{\widetilde{1}} \\ (1 - i(z^0 + ia))z^1 & 1 - i(z^{\widetilde{0}} + ia) \\ 1 + i(z^0 - ia) & -(1 + i(z^{\widetilde{0}} - ia))z^{\widetilde{1}} \\ (1 + i(z^0 + ia))z^1 & 1 + i(z^{\widetilde{0}} + ia) \end{pmatrix} \tag{19.58}$$

Like previously, we use the relations

$$
\begin{array}{lll}
z^0 = t - r + ia\cos\theta & , & z^1 = e^{i\varphi}\tan\frac{\theta}{2} \\
z^{\widetilde{0}} = t + r - ia\cos\theta & , & z^{\widetilde{1}} = \frac{r+ia}{r-ia}e^{-i\varphi}\tan\frac{\theta}{2}
\end{array}
\tag{19.59}
$$

to find the labels of L^μ rays

$$
s_1 := t - r \quad , \quad s_2 := \theta \quad , \quad s_3 := \varphi \tag{19.60}
$$

and assume the r parameter to indicate the points of the one ray. The coordinates z^α do not depend on r, remain invariant along the rays, therefore I keep them unchanged. Then we express only $z^{\widetilde{\alpha}}$ as functions of the proper L^μ ray coordinates (r, s_1, s_2, s_3)

$$
z^{\widetilde{0}} = s_1 + 2r - ia\cos\theta \quad , \quad z^{\widetilde{1}} = \frac{r+ia}{r-ia}e^{-i\varphi}\tan\frac{\theta}{2} \tag{19.61}
$$

and we find the rays in homogeneous coordinates $Y^{mi}(r)$.

In the context of the quadratic $CP(3)$ hypersurface, along the L^μ integral curves the one intersection point with the line is preserved constant and changes the second. For the N^μ integral curves the role of the intersection points are interchanged. Now using the relation

$$
\begin{array}{lll}
w_{11} = \frac{Y^{21}Y^{12}-Y^{11}Y^{22}}{Y^{01}Y^{12}-Y^{11}Y^{02}} & , & w_{12} = \frac{Y^{01}Y^{22}-Y^{21}Y^{02}}{Y^{01}Y^{12}-Y^{11}Y^{02}} \\
 & & \\
w_{21} = \frac{Y^{31}Y^{12}-Y^{11}Y^{32}}{Y^{01}Y^{12}-Y^{11}Y^{02}} & , & w_{12} = \frac{Y^{01}Y^{32}-Y^{31}Y^{02}}{Y^{01}Y^{12}-Y^{11}Y^{02}}
\end{array}
\tag{19.62}
$$

between the bounded projective $w \in U(2)$ and homogeneous Y^{ni} coordinates, we finally find the rays $w_L(r; s_1, s_2, s_3) \in U(2)$ in the complete bounded universe $U(2)$.

The intersection of the two $\mathbb{R}^4$ sheets in $U(2)$ coordinates can be computed by simply making the Cayley transformation of the cartesian form of the ring singularity. Then we find that in $(\tau, \rho, \sigma, \chi)$ coordinates the ring singularity (the caustic of the congruence) and its "tube" connecting the two sheets is

$$
\begin{array}{lll}
\sigma = \frac{\pi}{2} & , & R_0^2\frac{\sin^2\rho}{(\cos\tau+\cos\rho)^2} \leq a^2 \\
-\pi < \rho < \pi & , & -\pi < \tau < \pi
\end{array}
\tag{19.63}
$$

which apparently contains both rings of the two $\mathbb{R}^4$ copies.

In principle we can compute the explicit form of the L^μ ray tracing in $U(2)$, it is too complicated. From the cartesian coordinates we have

$$
\begin{array}{l}
x^0 = \frac{\sin\tau}{\cos\tau+\cos\rho} \\
x^1 + ix^2 = \frac{\sin\rho}{\cos\tau+\cos\rho}\sin\sigma\ e^{i\chi} = \sqrt{r^2+a^2}e^{-i\arctan\frac{a}{r}}\sin\theta e^{i\varphi} \\
x^3 = \frac{\sin\rho}{\cos\tau+\cos\rho}\cos\sigma = r\cos\theta \\
s := \frac{\sin\rho}{\cos\tau+\cos\rho}
\end{array}
\tag{19.64}
$$

which imply the following relations of the curve determining variables (s_1, s_2, s_3)

$$\begin{array}{l} s_1 := \frac{\sin\tau}{\cos\tau+\cos\rho} - r \\ s_2 := \tan\theta = \frac{r}{\sqrt{r^2+a^2}} \tan\sigma \\ s_3 := \varphi = \chi + \arctan\frac{a}{r} \\ \\ r^4 - [s^2 - a^2]r^2 - a^2 s^2 \cos^2\sigma = 0 \end{array} \tag{19.65}$$

and the convenient affine parameter of the congruence is r. The implied form of $w_L(r; s_1, s_2, s_3)$ is too complicated to be presented here.

In the context of conventional optics, such formations have been observed and they are called gaussian beams, which have complex centers. The electron seems to be such a gaussian beam with waist a in the entire flat universe.

19.5 Electron viewed from conformal infinity

Recall that the electron is a static axially symmetric LCR-structure relative to a precise grassmannian chart (coordinate system) with homogeneous and projective coordinates

$$\begin{array}{l} X^{mi} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ -irX_1 \end{pmatrix} \\ \det X_1 \neq 0 \quad , \quad r := iX_2X_1^{-1} \end{array} \tag{19.66}$$

where the linear (affine) transformation is

$$\begin{array}{l} \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ r' = iX_2'X_1'^{-1} = A_{22} r A_{11}^{-1} + iA_{21}A_{11}^{-1} \end{array} \tag{19.67}$$

and the Kerr polynomial (the quadric of $CP(3)$) of the electron has the form

$$K(Z^n) = Z^1Z^2 - Z^0Z^3 + 2aZ^0Z^1 \tag{19.68}$$

At the conformal coordinate chart

$$\begin{array}{l} X^{mi} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} i\widehat{r}X_2 \\ X_2 \end{pmatrix} \\ \det X_2 \neq 0 \quad , \quad \widehat{r} := -iX_1X_2^{-1} = r^{-1} \\ \widehat{x}^\mu := \frac{x^\mu}{\eta_{\rho\sigma}x^\rho x^\sigma} \end{array} \tag{19.69}$$

the affine transformation (Poincaré transformation) has the form

$$\begin{array}{l} \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ \widehat{r}' = -iX_1'X_2'^{-1} = A_{11}\widehat{r}A_{22}^{-1} - iA_{12}A_{22}^{-1} \end{array} \tag{19.70}$$

and the above Kerr polynomial is no longer static and axially symmetric relative to these coordinates. The caustic of the electron LCR-structure (the unit disk) will look like an exploding circle $(\widehat{x}^1)^2 + (\widehat{x}^2)^2 = \frac{a^2}{1+a^2}$.

19.6 Electromagnetic dressing of the electron

Because of the tetrad-Weyl symmetry, the LCR-manifold (determined by the tetrad $[\ell, m; n, \overline{m}]$) can define a class of metrics $[g_{\mu\nu}]$ and the corresponding class of self-dual 2-forms

$$
\begin{array}{l}
g_{\mu\nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu \\
\\
V_1 := \ell \wedge m \\
V_2 := n \wedge \overline{m} \\
V_3 := \ell \wedge n - m \wedge \overline{m}
\end{array}
\tag{19.71}
$$

which satisfy the relations

$$
\begin{array}{l}
dV_1 = [(2\varepsilon - \rho)n + (\tau - 2\beta)\overline{m}] \wedge V_1 \\
dV_2 = [(\mu - 2\gamma)\ell + (2\alpha - \pi)m] \wedge V_2 \\
dV_3 = 2[\mu\ell - \rho n - \pi m + \tau\overline{m}] \wedge V_3
\end{array}
\tag{19.72}
$$

where I have assumed the conditions $\kappa = \sigma = 0 = \lambda = \nu$, which define the LCR-structure. In this and the subsequent subsection I will explicitly show how the tetrad-Weyl symmetry is broken down and a precise metric and 2-form is imposed in the electron LCR-manifold.

In the case of LCR-manifolds compatible with the Minkowski metric $[\eta_{\mu\nu}]$, the tetrad-Weyl factors are trivially chosen. Recall that "flatness" is essentially defined algebraically. But in the case of the electron LCR-manifold it seems to be something more profound. It admits a closed but not exact 2-form $f(x)V_3$.

The LCR-tetrad

$$
\ell_\mu = L_\mu \quad , \quad m_\mu = M_\mu \quad , \quad n_\mu = N_\mu + \tfrac{h(r)}{2(r^2+a^2\cos^2\theta)}\ L_\mu
\tag{19.73}
$$

admits a generally complex function $f(x)$, such that

$$
\begin{array}{l}
\widehat{F} = \frac{c}{(r-ia\cos\theta)^2}(\ell \wedge n - m \wedge \overline{m}) = \frac{c}{(r-ia\cos\theta)^2}(L \wedge N - M \wedge \overline{M}) \\
d\widehat{F} = 0\ , \quad \forall x \notin \{caustic\} \\
\\
d[\mu\ell - \rho n - \pi m + \tau\overline{m}] = 0
\end{array}
\tag{19.74}
$$

where c is an arbitrary complex constant, and the last relation is the necessary and sufficient condition for the existence of such a self-dual 2-form. Notice that this definition continues to permit the tetrad-Weyl transformations with $\Lambda N = M\overline{M}$, that is the ordinary Weyl transformation, which will be fixed by the existence of a "gravitational" conserved current. It will be described in the next subsection.

We explicitly find

$$
\begin{array}{l}
\widehat{F} = \frac{c}{(r-ia\cos\theta)^2}[dt \wedge dr + ia\sin\theta dt \wedge d\theta - ia\sin\theta dr \wedge d\theta + \\
\qquad + a\sin^2\theta dr \wedge d\varphi + i(r^2+a^2)\sin\theta d\theta \wedge d\varphi]
\end{array}
\tag{19.75}
$$

Applying Stoke's theorem, we find the coefficient of the singularity at the caustic

$$\lim_{r\to\infty} \int_{t,r} \widehat{F} = 4ic\pi \tag{19.76}$$

We choose the arbitrary complex constant c, such that the implied real 2-form $F := \mathrm{Re}(\widehat{F})$ admits pure electric charge

$$dF = 0 \quad , \quad d*F = -*j_e$$

$$\begin{array}{l} F = \frac{q}{4\pi(r^2+a^2\cos^2\theta)^2}[(r^2-a^2\cos^2\theta)dt\wedge dr - 2a^2r\cos\theta\sin\theta dt\wedge d\theta + \\ \quad +2a^2r\cos\theta\sin\theta dr\wedge d\theta + a(r^2-a^2\cos^2\theta)\sin^2\theta dr\wedge d\varphi - \\ \quad -2ar(r^2+a^2)\cos\theta\sin\theta d\theta\wedge d\varphi = \\ \quad = d[\frac{qr}{4\pi(r^2+a^2\cos^2\theta)}(dt - dr - a\sin^2\theta d\varphi)] \end{array} \tag{19.77}$$

The electromagnetic potential

$$A = \frac{qr}{4\pi(r^2+a^2\cos^2\theta)}(dt - dr - a\sin^2\theta d\varphi) \tag{19.78}$$

is proportional to the retarded (causal) real covector ℓ_μ of the LCR-tetrad. Notice that the dimension of the potential A_μ is $\frac{[Q]}{[L]^2}$. Besides, there is a peculiar phenomenon that we have to understand. The existence of the closed 2-form $\widehat{F}$ is a "flat" LCR-structure property without intervention of the "curved" part of $n_\mu dx^\mu$. But the electromagnetic charge, appears in the gravitational dressing too.

In cartesian coordinates

$$\begin{array}{l} x^0 = t \\ x^1 = (r\cos\varphi + a\sin\varphi)\sin\theta \\ x^2 = (r\sin\varphi - a\cos\varphi)\sin\theta \\ x^3 = r\cos\theta \end{array} \tag{19.79}$$

$$r^4 - [(x^1)^2 + (x^2)^2 + (x^3)^2 - a^2]r^2 - a^2(x^3)^2 = 0$$

the electromagnetic potential takes the form

$$A = \frac{qr^3}{4\pi(r^4+a^2(x^3)^2)}(dx^0 - \frac{rx^1-ax^2}{r^2+a^2}dx^1 - \frac{rx^2+ax^1}{r^2+a^2}dx^2 - \frac{x^3}{r}dx^3) \tag{19.80}$$

which is singular at the ring

$$x^0 = t \quad , \quad (x^1)^2 + (x^2)^2 = a^2 \quad , \quad x^3 = 0 \tag{19.81}$$

$\forall x^0 \in \mathbb{R}$. Notice that it is not a point.

Now let me make a fundamental remark. The field strength 2-form is a singular function at the caustic. We cannot compute the classical electromagnetic energy, because it diverges at the caustic. In the context of quantum electrodynamics this problem is solved in the context of Schwartz generalized functions. Recall that singular functions are representatives of Schwartz distributions, if they are derivatives of locally integrable functions. The above electromagnetic potential A_μ has locally integrable components, because

$$\begin{aligned}
&\int_0^\infty dr \int d\theta \tfrac{r^2 \sin\theta}{r^2+a^2\cos^2\theta}\phi(r) = -\lim_{\varepsilon\to+0}\int_\varepsilon^\infty dr\tfrac{r}{a}\phi(r)\int dc' \tfrac{1}{1+c'^2} = \\
&\qquad = \lim_{\varepsilon\to+0}\int_\varepsilon^\infty dr \tfrac{2r}{a}\phi(r)\arctan\tfrac{a}{r} \qquad (19.82)\\
&c' := \tfrac{a}{r}\cos\theta
\end{aligned}$$

($\phi(r)$ is a test function) is finite. It permit us to make computations with the field strength and all its derivatives as long as we respect the rules of Schwartz distributions. When we multiply generalized functions, we must be careful with their wavefront singularities. This is the origin of the infinity problems we have with the conserved electromagnetic energy (=square of field strength), which are solved by normal ordering and "renormalizability" procedure in quantum electrodynamics.

19.7 Gravitational dressing of the electron

The electromagnetic conserved current restricts the tetrad-Weyl symmetry down to the ordinary Weyl symmetry, which is a symmetry of the electromagnetic field. The further restriction will be imposed from a conserved current in the class of metrics $[g_{\mu\nu}]$. Taking into account that the conformal tensor is Weyl invariant, the conserved current has to emerge from the Bianchi part of the curvature.

It is well known in general relativity, that the Kerr-Schild ansatz in cartesian coordinates linearizes the Einstein tensor. In cartesian coordinates

$$\begin{aligned}
&x^0 = t\\
&x^1 = (r\cos\varphi + a\sin\varphi)\sin\theta\\
&x^2 = (r\sin\varphi - a\cos\varphi)\sin\theta\\
&x^3 = r\cos\theta\\
&Jacobian = (r^2+a^2\cos^2\theta)\sin\theta
\end{aligned} \qquad (19.83)$$

we chose the tetrad-Weyl factors such that a representative of the electron class of metrics takes the form

$$\begin{aligned}
&g_{\mu\nu} = \eta_{\mu\nu} + \phi\ L_\mu L_\nu\\
&\phi = \tfrac{-2Mr^3+e^2r^2}{r^4+a^2(x^3)^2}\\
&L_\mu dx^\mu = \ dx^0 - \tfrac{(rx^1-ax^2)}{r^2+a^2}dx^1 - \tfrac{(rx^2+ax^2)}{r^2+a^2}dx^2 - \tfrac{x^3}{r}dx^3\\
&r^4 - [(x^1)^2+(x^2)^2+(x^3)^2-a^2]r^2 - a^2(x^3)^2 = 0
\end{aligned} \qquad (19.84)$$

The Ricci tensor and scalar curvature take the linear form

$$\begin{array}{l} R^{\mu}_{\nu}=\frac{1}{2}[\partial^{\mu}\partial_{\rho}h^{\rho}_{\nu}+\partial_{\nu}\partial_{\rho}h^{\rho\mu}-\partial^{\mu}\partial_{\nu}h-\eta^{\rho\sigma}\partial_{\rho}\partial_{\sigma}h^{\mu}_{\nu}] \\ R=\partial_{\nu}\partial_{\mu}h^{\nu\mu}-\eta^{\rho\sigma}\partial_{\rho}\partial_{\sigma}h \\ \\ g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \quad , \quad h:=\eta^{\mu\nu}h_{\mu\nu} \\ h_{\mu\nu}:=\frac{-2Mr^3+e^2r^2}{r^4+a^2(x^3)^2}L_{\mu}L_{\nu} \end{array} \tag{19.85}$$

Hence the Einstein tensor of the electron gravitational dressing is linear in $h_{\mu\nu}$ and the corresponding Bianchi identity leads to a conserved current for energy-momentum and angular momentum, which breaks the Weyl symmetry. Besides, the implied coupling constant of the gravitational dressing is not dimensionless.

In the unbounded realization of the classical domain, the flat universe is locally the real axis $\mathbb{R}^4$ and the computations are simple because the flat metric is constant, but we miss the global structure of the entire universe $U(2)$. The second $\mathbb{R}^4$-sheet is beyond infinity. But in the case of the electron, the tracing of LCR-rays indicates that it folds with infinity back to the local singularity disk (the location of the electron). Therefore the global picture of the universe is necessary to describe the electron and the positron as well defined LCR-manifolds.

We have already showed, that the induced metric implied by the bounded ambient "flat" Kaehler manifold is

$$\begin{array}{l} ds^2_B=(d\tau)^2-(d\rho)^2-\sin^2\rho(d\chi)^2-\sin^2\rho\sin^2\chi(d\psi)^2=:\widehat{\eta}_{\mu\nu}dx^{\mu}dx^{\mu} \\ \tau\in(0,2\pi) \quad , \quad \rho\in[0,2\pi) \quad , \quad \theta\in[0,\pi] \quad , \quad \varphi\in[0,2\pi) \end{array} \tag{19.86}$$

This metric $\widehat{\eta}_{\mu\nu}$ is a conformal transformation of the Minkowski metric $\eta_{\mu\nu}$. We precisely have

$$ds^2_U=\frac{1}{(\cos\tau+\cos\rho)^2}ds^2_B \tag{19.87}$$

Hence, the Einstein equations are no longer linear and the conservation law is not valid. The breaking mechanism does not work. Therefore it is essential to work in one from the two $\mathbb{R}^4$-sheets of the universe, where even gravitational dressing admits conserved quantities because of its linearized equation.

In ordinary general relativity the conservation of energy-momentum and angular momentum is derived from pseudotensors. The Landau-Lifshitz pseudotensor is

$$T^{\mu\nu}_{LL}:=\frac{c^4}{16\pi k}\frac{\partial^2}{\partial x^{\rho}\partial x^{\sigma}}[(-g)(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\rho\nu})] \tag{19.88}$$

which must be used in cartesian coordinates. In PCFT these coordinates are well defined as the real part of the complex projective coordinates of $G(4,2)$. It is evident that it breaks Weyl symmetry

$$\begin{array}{l} g_{\mu\nu}\to\Phi^2g_{\mu\nu} \\ \\ g^{\mu\nu}\to\frac{1}{\Phi^2}g^{\mu\nu} \quad , \quad g\to\Phi^8g \end{array} \tag{19.89}$$

If we replace the LCR-tetrad, we find

$$T^{\mu\nu}_{LL}:=\frac{c^4}{16\pi k}\frac{\partial^2}{\partial x^{\rho}\partial x^{\sigma}}[(-g)(V_1^{\sigma\mu}V_2^{\nu\rho}+V_2^{\sigma\mu}V_1^{\nu\rho}+\frac{1}{2}V_3^{\sigma\mu}V_3^{\nu\rho}+c.c.)] \tag{19.90}$$

where

$$
\begin{aligned}
&V_1^{\sigma\mu} := \ell^\sigma m^\mu - \ell^\mu m^\sigma \quad , \quad V_2^{\sigma\mu} := n^\sigma \overline{m}^\mu - n^\mu \overline{m}^\sigma \\
&V_3^{\sigma\mu} := \ell^\sigma n^\mu - \ell^\mu n^\sigma - m^\sigma \overline{m}^\mu + m^\mu \overline{m}^\sigma
\end{aligned}
\tag{19.91}
$$

It is not generally covariant.

19.8 LCR-rays of the (curved) electron LCR-manifold

From the relation between the projective and homogeneous coordinates

$$
\begin{aligned}
r^0 &= i\tfrac{(X^{01}X^{32}-X^{31}X^{02})+(X^{21}X^{12}-X^{11}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})} \\
r^1 &= i\tfrac{(X^{11}X^{32}-X^{31}X^{12})+(X^{21}X^{02}-X^{01}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})} \\
r^2 &= i\tfrac{(X^{11}X^{32}-X^{31}X^{12})-(X^{21}X^{02}-X^{01}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})} \\
r^3 &= i\tfrac{(X^{01}X^{32}-X^{31}X^{02})-(X^{21}X^{12}-X^{11}X^{22})}{2(X^{01}X^{12}-X^{11}X^{02})}
\end{aligned}
\tag{19.92}
$$

we make the substitutions

$$
X^{mi} = \begin{pmatrix} 1 & -z^{\widetilde{1}} \\ z^1 & 1 \\ -i(z^0 - ia) & i(z^{\widetilde{0}} - ia)z^{\widetilde{1}} \\ -i(z^0 + ia)z^1 & -i(z^{\widetilde{0}} + ia) \end{pmatrix}
\tag{19.93}
$$

and we find

$$
\begin{aligned}
r^0 &= \tfrac{z^0+z^{\widetilde{0}}}{2} \quad , \quad r^1 = \tfrac{(z^{\widetilde{0}}-z^0)(z^1+z^{\widetilde{1}})}{2(1+z^1 z^{\widetilde{1}})} \\
r^2 &= \tfrac{(z^{\widetilde{0}}-z^0)(z^1-z^{\widetilde{1}})}{2(1+z^1 z^{\widetilde{1}})} \quad , \quad r^3 = \tfrac{(z^0+z^{\widetilde{0}})(1-z^1 z^{\widetilde{1}})}{2(1+z^1 z^{\widetilde{1}})}
\end{aligned}
\tag{19.94}
$$

In the case of the ℓ^μ congruence we find

$$
\begin{aligned}
&r^a(r) := x_\ell^a(r) + iy_\ell^a(r) \\
&r^0 = u + r + M\ln\tfrac{\Delta}{r_1} + \tfrac{e^2-2M^2}{\Theta}\arctan\tfrac{\Theta}{r-M} \\
&r^1 = \frac{(r - ia\cos\theta + M\ln\frac{\Delta}{r_1} + \frac{e^2-2M^2}{\Theta}\arctan\frac{\Theta}{r-M})(e^{i\varphi} + r_2(\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}\ e^{-i\varphi})\tan\frac{\theta}{2}}{1 + r_2(\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}\ \tan^2\frac{\theta}{2}} \\
&r^2 = \frac{(r - ia\cos\theta + M\ln\frac{\Delta}{r_1} + \frac{e^2-2M^2}{\Theta}\arctan\frac{\Theta}{r-M})(e^{i\varphi} - (\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}\ e^{-i\varphi})\tan\frac{\theta}{2}}{1 + r_2(\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}\ \tan^2\frac{\theta}{2}} \\
&r^3 = \frac{(u + r + M\ln\frac{\Delta}{r_1} + \frac{e^2-2M^2}{\Theta}\arctan\frac{\Theta}{r-M})(1 - (\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}\ \tan^2\frac{\theta}{2})}{1 + r_2(\frac{r-M+i\Theta}{r-M-i\Theta})^{\frac{a}{\Theta}}\ \tan^2\frac{\theta}{2}} \\
&\Delta := r^2 + a^2 - 2Mr + e^2 \quad , \quad \Theta := \sqrt{a^2 + e^2 - M^2}
\end{aligned}
\tag{19.95}
$$

where u, θ, φ are constants along the corresponding integral curve.

We have already shown that the LCR-manifold is defined as a special real submanifold of an ambient complex manifold which admits a class of Kaehler metrics determined by the $\det(\rho_{ij})$. It is a symplectic manifold with a closed symplectic form which vanishes in the LCR-manifold, i.e. it is a lagrangian submanifold with the corresponding class of metrics. The above relations (19.95) determine the embedding of this lagrangian submanifold in the canonical coordinates of the phase space.

In the case of the n^μ congruence, the constants along its integral curve are

$$\begin{aligned}
&v := t + r + 2M \ln \tfrac{\Delta}{r_0} + \tfrac{2(e^2-2M^2)}{\Theta} \arctan \tfrac{\Theta}{r-M} \\
&\theta' = \theta \\
&e^{-i\varphi'} = r_2 \left(\tfrac{r-M+i\Theta}{r-M-i\Theta}\right)^{\frac{a}{\Theta}} e^{-i\varphi}
\end{aligned} \tag{19.96}$$

The form of the curves $r^a := x_n^a(r) + i y_n^a(r)$ is found through the appropriate substitutions.

19.9 The positron

The positron LCR-structure is the conjugate of the electron one

$$\begin{aligned}
&z'^0 = \overline{z^0} = t - r - ia\cos\theta \quad , \quad z'^1 = \overline{z^1} = e^{-i\varphi} \tan \tfrac{\theta}{2} \\
&z'^{\widetilde{0}} = \overline{z^{\widetilde{0}}} = t + r + ia\cos\theta + 2M \ln \tfrac{|\Delta|}{r_1} + \tfrac{2(e^2-2M^2)}{\Theta} \arctan \tfrac{\Theta}{r-M} \\
&z'^{\widetilde{1}} = \overline{z^{\widetilde{1}}} = r_2 \left(\tfrac{r-M-i\Theta}{r-M+i\Theta}\right)^{\frac{a}{\Theta}} e^{i\varphi} \tan \tfrac{\theta}{2}
\end{aligned} \tag{19.97}$$

which has the LCR-tetrad

$$\ell'_\mu = \ell_\mu \quad , \quad m'_\mu = \overline{m}_\mu \quad , \quad n'_\mu = n_\mu \quad , \quad \overline{m}'_\mu = m_\mu \tag{19.98}$$

which has the same gravitational dressing but opposite charge electromagnetic dressing.

In order to geometrically distinguish the positron from the electron we have to consider the (unbounded) two $\mathbb{R}^4$-sheets realization, where the ℓ^μ and n^μ congruences have opposite directions passing through the ring singularity. In a given $\mathbb{R}^4$-sheet, assuming that the electron is the LCR-manifold with the ℓ^μ congruence outgoing and the n^μ congruence ingoing, the positron is the LCR-manifold with the ℓ'^μ congruence outgoing with opposite twist a and the n'^μ congruence ingoing with opposite twist too. Hence the outgoing (retarded) and ingoing (advanced) character of their LCR-rays is preserved.

20 "ACCELERATED" ELECTRON LCR-STRUCTURE

Algebraic surfaces may be defined either implicitly through polynomials or explicitly through holomorphic functions. We saw that the free electron hypersurface of $CP(3)$ is either implicitly determined by the Poincaré covariant quadric fixed by the 4-velocity v^a of the free electron or the linear trajectory $\xi^a(\tau) = v^a\tau + s^a$ with $v^a v^b \eta_{ab} = 1$ and s^a a generally complex vector related to the spacetime location and the spin of the electron. In this section we will use the general complex trajectory to describe the generally moving electron.

In the case of a ruled LCR-structure determined by one trajectory $\xi^a(\tau)$ i.e.

$$X = \begin{pmatrix} \lambda^1 & \lambda^2 \\ -i\xi_a(\tau_1)\sigma^a\lambda^1 & -i\xi_a(\tau_2)\sigma^a\lambda^2 \end{pmatrix} \tag{20.1}$$

$z^0, \widetilde{z^0}$ are the two solutions and $\lambda^{A1}, \lambda^{A2}$ are the corresponding projective solutions of the following equations

$$
\begin{aligned}
&\det[(r_a - \xi_a(\tau))\sigma^a] = 0 \quad , \quad \tau_{1,2} = \xi^0 = r^0 \mp \sqrt{(r^i - \xi^i(\tau_{1,2}(r^b)))^2} \\
&(r_a - \xi_a(z^0))\sigma^a_{A'A}\lambda^{A1} = 0 = (r_a - \xi_a(\widetilde{z^0}))\sigma^a_{A'A}\lambda^{A2}
\end{aligned}
\tag{20.2}
$$

Notice that if the two roots $\tau_{1,2}$ are different the corresponding λ^1 and λ^2 are also different and they provide the natural "retarded" and "advanced" intersection points of the line r^a with two sheets of the algebraic surface of $CP(3)$.

In the zero gravity approximation $X^\dagger E X = 0$, i.e. $r^a = x^a = \overline{x^a}$, the LCR-tetrad is compatible with the Minkowski metric. Hence the spinors $\lambda^{Aj} = (1, \zeta_j)^\top$ may be projectively computed via the above relations. One can easily see that the Kerr functions, which determine the ℓ^μ and n^μ geodetic and shear-free null congruences are found after the elimination of τ_j from the following equivalent (because of the vanishing of the determinant) two pairs of relations

$$
\begin{aligned}
&\left[(x^1 - ix^2) - (\xi^1(\tau_j) - i\xi^2(\tau_j))\right]\zeta_j - \left[(x^0 - x^3) - (\xi^0(\tau_j) - \xi^3(\tau_j))\right] = 0 \\
&\left[(x^0 + x^3) - (\xi^0(\tau_j) + \xi^3(\tau_j))\right]\zeta_j - \left[(x^1 + ix^2) - (\xi^1(\tau_j) + i\xi^2(\tau_j))\right] = 0
\end{aligned}
\tag{20.3}
$$

i.e.

$$
\zeta_j = \frac{(x^0 - x^3) - (\xi^0(\tau_j) - \xi^3(\tau_j))}{(x^1 - ix^2) - (\xi^1(\tau_j) - i\xi^2(\tau_j))} = \frac{(x^1 + ix^2) - (\xi^1(\tau_j) + i\xi^2(\tau_j))}{(x^0 + x^3) - (\xi^0(\tau_j) + \xi^3(\tau_j))}
\tag{20.4}
$$

and the corresponding flat null LCR-tetrad is

$$
\begin{aligned}
\ell^\mu &= N_1\left(1 + \zeta_1\overline{\zeta_1}\ ,\ \zeta_1 + \overline{\zeta_1}\ ,\ -i(\zeta_1 - \overline{\zeta_1})\ ,\ 1 - \zeta_1\overline{\zeta_1}\right) \\
n^\mu &= N_2\left(1 + \zeta_2\overline{\zeta_2}\ ,\ \zeta_2 + \overline{\zeta_2}\ ,\ -i(\zeta_2 - \overline{\zeta_2})\ ,\ 1 - \zeta_2\overline{\zeta_2}\right) \\
m^\mu &= N_3\left(1 + \zeta_1\overline{\zeta_2}\ ,\ \zeta_1 + \overline{\zeta_2}\ ,\ -i(\zeta_1 - \overline{\zeta_2})\ ,\ 1 - \zeta_1\overline{\zeta_2}\right)
\end{aligned}
\tag{20.5}
$$

As expected, the normalization factors cannot be computed and they have to be imposed.

In the considered case of zero gravity approximation the imbedding relations of the LCR-manifold can be easily written down and subsequently the LCR-tetrad can be computed as functions of the structure coordinates. Assuming the notation

$$
X = \begin{pmatrix} \lambda & \widetilde{\lambda} \\ -i\xi_a\sigma^a\lambda & -i\widetilde{\xi}_a\sigma^a\widetilde{\lambda} \end{pmatrix}
\tag{20.6}
$$

where $\xi_a(z^0)$ and $\widetilde{\xi}_a(\widetilde{z^0})$ are two independent trajectories for every column, the embedding relations are

$$
\rho_{ij} = X^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X = \begin{pmatrix} i(\overline{\xi_a} - \xi_a)\lambda^\dagger\sigma^a\lambda & i(\overline{\xi_a} - \widetilde{\xi}_a)\lambda^\dagger\sigma^a\widetilde{\lambda} \\ -i(\xi_a - \overline{\widetilde{\xi}_a})\widetilde{\lambda}^\dagger\sigma^a\lambda & i(\overline{\widetilde{\xi}_a} - \widetilde{\xi}_a)\widetilde{\lambda}^\dagger\sigma^a\widetilde{\lambda} \end{pmatrix} = 0
\tag{20.7}
$$

where z^0 and $z^{\widetilde{0}}$ are the variables of the two independent trajectories and

$$\lambda = \begin{pmatrix} 1 \\ z^1 \end{pmatrix} \quad , \quad \widetilde{\lambda} = \begin{pmatrix} -z^{\widetilde{1}} \\ 1 \end{pmatrix} \tag{20.8}$$

are the other structure coordinates. Following the conventional procedure, the LCR-tetrad can be found.

The integral curves (LCR-rays) of the retarded ℓ^μ congruence are labeled by the three independent functions $\mathrm{Re}(z^0)$, $\mathrm{Re}(z^1)$ and $\mathrm{Im}(z^1)$ of the structure coordinates z^β. The first is the wavefront surface. Hence, the algebraic trajectory $\xi^a(z^0)$ is the movement of the source of retarded rays in the grassmannian space $G(4,2)$. In the case of the advanced n^μ rays the $z^{\widetilde{\beta}}$ structure coordinates are considered.

20.1 Real trajectory and Lienard-Wiechert potentials

In the electron LCR-structure, the spin of the electron appears as non-vanishing relative invariants. Hence, if we neglect the spin, the relative invariants vanish, i.e. the LCR-structure is degenerate. Then the LCR-tetrad (up to a factor) is

$$\begin{array}{ll} \ell = dz^0 = d\tau_1 \quad , & m = dz^1 = d\zeta_1 \\ n = dz^{\widetilde{0}} = d\tau_2 \quad , & \overline{m} = dz^{\widetilde{1}} = d\zeta_2 \end{array} \tag{20.9}$$

They are found to be

$$\begin{array}{l} \ell = \frac{x_\mu - \xi_\mu(z^0)}{(x^a - \xi^a(z^0))\dot{\xi}_a(z^0)} dx^\mu \quad , \\ m = \frac{-1}{(x^0+x^3)-(\xi^0(\tau_j)+\xi^3(\tau_j))}[-\zeta_1(dx^0+dx^3)+dx^1+idx^2)+(\zeta_1(\dot{\xi}^0+\dot{\xi}^3)-(\dot{\xi}^1+i\dot{\xi}^2))\ell] \\ n = \frac{x_\mu - \xi_\mu(z^{\widetilde{0}})}{(x^a - \xi^a(z^{\widetilde{0}}))\dot{\xi}_a(z^{\widetilde{0}})} dx^\mu \end{array} \tag{20.10}$$

I point out that the tetrad gives the closed self-dual 2-form $J = \frac{1}{2} J_{\mu\nu} dx^\mu \wedge dx^\nu$, but it has to be multiplied with the appropriate tetrad-Weyl factors in order to give the Minkowski metric.

The self-dual 2-form and the retarded Lienard-Wiechert potential is

$$\begin{array}{l} J = 2dz^0 \wedge dz^{\widetilde{0}} - 2dz^1 \wedge dz^{\widetilde{1}} \\ A^\mu = \frac{e\dot{\xi}^\mu(z^0)}{4\pi(x^a - \xi^a(z^0))\dot{\xi}_a(z^0)} \end{array} \tag{20.11}$$

which have a singularity along the trajectory $\xi^a(\tau)$, which is interpreted as the trajectory of the charge. In the case of a spinless charge at rest $\xi^a(\tau) = (\tau, 0, 0, 0)^\top$, we simply find the electric potential and vanishing magnetic potential.

The electromagnetic quantum mode, the photon, is the signal singularity, which can occur at the retarded (or advanced) wave front $z^0 = 0$ and it moves with the velocity of light. This is implied by the Cauchy problem applied on the electromagnetic partial differential equations. Notice that the LCR-structure contains an additional singularity, the trajectory (the intersection of the two sheets of $CP(3)$), which describes the movement of a particle with lower than light velocity. This last singularity emerges at the points of the LCR-manifold where the LCR-tetrad is not linearly independent.

20.2 Expansion in 1/c

In this subsection, we continue to neglect gravity. Besides we normalize the dummy (generally complex) parameter τ of the trajectory $\xi^a(\tau)$ with the condition $\xi^0(\tau) = \tau$. Considering analytic trajectories, we may separate them to their real and imaginary parts $\xi^a(\tau) = \xi^a_R(\tau) + i\xi^a_I(\tau)$. That is the coefficients of their Taylor expansion in τ, of $\xi^a_R(\tau)$ and $\xi^a_I(\tau)$ are real. Then we have the following structure coordinates

$$\begin{array}{l} z'^0 = t - \frac{1}{c}\sqrt{(x^i - \xi^i_R(z'^0) - i\xi^i_I(z'^0))^2} \\ z'^{\widetilde{0}} = t + \frac{1}{c}\sqrt{(x^i - \xi^i_R(z'^{\widetilde{0}}) - i\xi^i_I(z'^{\widetilde{0}}))^2} \end{array} \tag{20.12}$$

where I substituted $x^0 = ct$, in order to make appear the velocity of light, and after I divided z^0 and $z^{\widetilde{0}}$, in order to facilitate the $\frac{1}{c}$ expansion. The physicist should notice here that the existence of the constant c (which accommodates the different time and distance units) is imposed be the addition of time and distance quantities.

The up to 1st order approximation implies

$$\begin{array}{l} z'^0 \simeq t - \frac{1}{c}\sqrt{(x^i - \xi^i_R(t) - i\xi^i_I(t))^2} \\ z'^{\widetilde{0}} \simeq t + \frac{1}{c}\sqrt{(x^i - \xi^i_R(t) - i\xi^i_I(t))^2} \end{array} \tag{20.13}$$

The trajectory (source) singularity of the LCR-structure occurs at the space curve

$$\begin{array}{c} (x^i - \xi^i_R(t))^2 = (\xi^j_I(t))^2 \\ \\ \sum\limits_{i=1}^{3} (x^i - \xi^i_R(t))\xi^i_I(t) = 0 \end{array} \tag{20.14}$$

In order to find the form of a current concentrated on this circumference s, I first consider the spheroidal coordinates

$$\begin{array}{l} x^i - \xi^i_R(t) = \widehat{c}^i\sqrt{r^2+a^2}\cos\varphi\sin\theta + \widehat{d}^i\sqrt{r^2+a^2}\sin\varphi\sin\theta + \widehat{\xi_I}^i\, r\cos\theta \\ a^2 = \xi^2_I = \sum\limits_{i=1}^{3}\xi^i_I\xi^i_I \quad , \quad \widehat{\xi_I}^i = \frac{\xi^i_I}{a} \end{array} \tag{20.15}$$

where $\widehat{c}^i$ and $\widehat{d}^i$ are unit vectors perpendicular to ξ^i_I. The jacobian of the transformation is

$$J = (r^2 + a^2\cos^2\theta)\sin\theta \tag{20.16}$$

Looking for a solution of the laplacian not defined on a curve s, we write

$$\partial^2 A^\mu(x) = j^\mu(x) \tag{20.17}$$

where j^μ has support on s. The Green's function is

$$\partial^2 G^\mu_\nu(x;x') = \delta^\mu_\nu \delta(x^\rho - x'^\rho) \tag{20.18}$$

in cartesian coordinates. It gives the solution

$$A^\mu(x) = \int\limits_V G^\mu_\nu(x;x') j^\nu(x') dx' = \int\limits_s G^\mu_\nu(x;x'(s)) j^\nu(x'(s)) dc \tag{20.19}$$

In the case of a scalar field we have

$$\begin{aligned}
&j(x')dx' = j_c(\varphi)\tfrac{\delta(r)\delta(\theta-\frac{\pi}{2})}{J} J dr d\theta d\varphi \\
&A(x) = \int\limits_V G(x;x') j(x') dx' = \int\limits_s G(x;x'(\varphi)) j_s(\varphi)) d\varphi
\end{aligned} \tag{20.20}$$

If we consider every point of the curve as a moving particle along the trajectory we will have

$$\begin{aligned}
&y^\mu = (t,\ y^i(t)) \\
&y^i = \xi^i_R(t) + \widehat{c}^i(t) a(t) \cos\varphi + \widehat{d}^i(t) a(t) \sin\varphi \\
&j^\mu = \dot{y}^\mu \delta(x^i - y^i(t))
\end{aligned} \tag{20.21}$$

I think this approximation gives the "classical" meaning of electron spin.

21 MASSLESS LCR-MANIFOLD

Instead of the riemannian structure $g_{\mu\nu}$ of Einstein's gravity, PCFT assumes the LCR-structure as fundamental as the fundamental structure of nature. Even at the "classical level" the elementary particles are derived as distributional solitons. In the present computational procedure, they correspond to algebraic hypersurfaces of $CP(3)$. The classical trajectory of the elementary particle is a manifestation of the complex trajectory in the grassmannian manifold $G(4,2)$ of the ruled surfaces $X^n(\tau,s) = X^{n1}(\tau) + sT^n(\tau)$. The pair massive-massless lepton is a manifestation of ruled surfaces with non-vanishing and vanishing gaussian curvatures, which are called developable and characterized by the property $T^n(\tau) = \frac{dX^{n1}(\tau)}{d\tau}$. In the present section we will specify this general framework in the case of the electron neutrino.

It is computationally easier to first look for a LCR-structure compatible with a minkowskian class $[\eta_{\mu\nu}]$ of metrics (a flatprint in the terminology of general relativity) and after applying a Kerr-Schild ansatz to find a curved candidate. So we look for a Kerr polynomial (11.3) with $\det p = 0$, which is automorphic relative to the z-rotation. No rank-3 quadratic surface survives this condition.

For every helicity $[E := p^0 = \pm p^3]$ of the neutrino LCR-structure, I only find the rank-2 union of the following two planes

$$\begin{aligned} &[E = -p^3]: \quad (X^1 + aX^3)X^0 = 0 \\ &[E = +p^3]: \quad X^1(X^0 + bX^2) = 0 \end{aligned} \tag{21.1}$$

in the frame with $p^1 + ip^2 = 0$. Note that both cases are deformations of the "cartesian" LCR-structure.

Let us consider the first case. In the affine space

$$X^{ni} = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ X^{21} & X^{22} \\ X^{31} & X^{32} \end{pmatrix} =: \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad , \quad \det X_1 \neq 0 \tag{21.2}$$

of the grassmannian $G(4,2)$, the Kerr polynomials of the left and right columns are

$$X^{11} + aX^{31} = 0 \quad , \quad X^{02} = 0 \tag{21.3}$$

respectively. Then in $CP(3)$, the intersections of the line x with the first and second planes are

$$\begin{aligned} &x := iX_2X_1^{-1} \\ &-a(x^1 + ix^2)X^{01} + [a(x^0 + x^3) + i]X^{11} = 0 \\ &X^{02} = 0 \end{aligned} \tag{21.4}$$

Notice that this LCR-structure is regular in the present affine chart ($\det(X_1) \neq 0$), because $[a(x^0 + x^3) + i] \neq 0$.

The convenient structure coordinates are

$$X^{ni} = \begin{pmatrix} 1 & 0 \\ az^1 & 1 \\ -iz^0 & z^{\widetilde{1}}(1 + iaz^{\widetilde{0}}) \\ -z^1 & -iz^{\widetilde{0}} \end{pmatrix} \tag{21.5}$$

The LCR-structure conditions ($X^\dagger E_U X = 0$) are

$$\begin{aligned} &\tfrac{z^0 - \overline{z^0}}{2i} - az^1\overline{z^1} = 0 \\ &z^{\widetilde{1}} - \overline{z^1} = 0 \quad , \quad \tfrac{z^{\widetilde{0}} - \overline{z^{\widetilde{0}}}}{2i} = 0 \end{aligned} \tag{21.6}$$

and the structure coordinates are

$$\begin{aligned} &z^0 = x^0 - x^3 - a\tfrac{(x^1)^2 + (x^2)^2}{a(x^0 + x^3) + i} \quad , \quad z^1 = -\tfrac{X^{31}}{X^{01}} = \tfrac{x^1 + ix^2}{a(x^0 + x^3) + i} \\ &z^{\widetilde{0}} = x^0 + x^3 \quad , \quad z^{\widetilde{1}} = \tfrac{x^1 - ix^2}{a(x^0 + x^3) - i} = \overline{z^1} \end{aligned} \tag{21.7}$$

The flat LCR-tetrad and its differential conditions are found following the general computational rules

$$
\begin{array}{l}
L = du - ia\overline{z^1}dz^1 + iaz^1 d\overline{z^1} \quad , \quad M = dz^1 \quad , \quad N = dv \\
u := \frac{z^0 + \overline{z^0}}{2} \quad , \quad v := \frac{z^{\tilde{0}} + \overline{z^{\tilde{0}}}}{2} \\
dL = 2iaM \wedge \overline{M} \quad , \quad dM = 0 \quad , \quad dN = 0
\end{array}
\tag{21.8}
$$

The integral curves $x^\mu_\ell(s)$ of the causal vector $\ell^\mu \partial_\mu$ are found using the definition of the projective coordinates and the fact that z^0 and z^1 are constant along the curves, because $\ell^\mu \partial_\mu z^0 = 0 = \ell^\mu \partial_\mu z^1$. We assume the ray (affine) parameter $s := z^{\tilde{0}} = x^0 + x^3$ and the ray labels $s_1 = \mathrm{Re}(z^0)$, $s_2 = \mathrm{Re}(z^1)$, and $s_3 = \mathrm{Im}(z^1)$. Then the jacobian is

$$
ds \wedge ds_1 \wedge ds_2 \wedge ds_3 = \frac{8}{a^2(x^0+x^3)^2+1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3
\tag{21.9}
$$

Hence the caustic of the causal rays is at infinity as we algebraically found it above.

The coordinate singularity is hidden at infinity, like in the "cartesian light-cone" LCR-structure (18.23). In order to "see" it, we have to work in the patch $\det X_2 \neq 0$, where

$$
\begin{array}{l}
x' := -iX_1(X_2)^{-1} = x^{-1} \\
-(x'^1 + ix'^2)X^{21} + [(x'^0 + x'^3) - ia]X^{31} = 0 \\
(x'^0 - x'^3)X^{22} - (x'^1 - ix'^2)X^{32} = 0
\end{array}
\tag{21.10}
$$

The singularity occurs at

$$
\begin{array}{c}
\det X_2 = 0 \\
\Downarrow \\
x'^0 - x'^3 = 0 \quad , \quad x'^1 = 0 = x'^2
\end{array}
\tag{21.11}
$$

We may have a global view of the LCR-manifold in the "bounded realization" coordinate chart Y^{ni} is given by

$$
\begin{array}{l}
X^0 = \frac{1}{\sqrt{2}}(Y^0 + Y^2) \quad , \quad X^1 = \frac{1}{\sqrt{2}}(Y^1 + Y^3) \\
X^2 = \frac{1}{\sqrt{2}}(Y^0 - Y^2) \quad , \quad X^3 = \frac{1}{\sqrt{2}}(Y^1 - Y^3)
\end{array}
\tag{21.12}
$$

where the two planes take the form

$$
\begin{array}{l}
(a+1)Y^{11} + (1-a)Y^{31} = 0 \quad , \quad Y^{02} + Y^{22} = 0 \\
\qquad\qquad\qquad \Downarrow \\
(1-a)w_{21}Y^{01} + [(a+1) + (1-a)w_{22}]Y^{11} = 0 \\
(1 + w_{11})Y^{02} + w_{12}Y^{12} = 0
\end{array}
\tag{21.13}
$$

and the singular surface ($\det Y_1 = 0$) is

$$
\begin{array}{l}
[w_{11}w_{22} - w_{12}w_{21} + 1 + w_{11} + w_{22}] + a[-w_{11}w_{22} + w_{12}w_{21} + 1 + w_{11} - w_{22}] = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow \\
\cos\tau + \cos\rho = 0 \quad , \quad \sin\tau + \sin\rho\cos\sigma = 0 \\
\tau \in (0, 2\pi) \quad , \quad \rho \in [0, 2\pi) \quad , \quad \sigma \in [0, \pi]
\end{array}
\tag{21.14}
$$

Notice the difference between the "cartesian" LCR-manifold and the present "massless" LCR-manifold. The former is singular in the entire null infinity (scri+ and scri-)(18.32), while the present is singular at a part of it. This subset must satisfy

$$
(\sin\rho)(\sin\sigma) = 0 \quad ; \quad \rho \in [0, 2\pi), \quad \sigma \in [0, \pi]
\tag{21.15}
$$

For $\sigma = 0$, we find the $\tau = 2\pi - \rho$ line and for $\sigma = \pi$, we find the $\tau = \rho$ curve.

In order to avoid the coordinate singularities we may look at the causal rays $x^\mu_\ell(s)$ of $\ell^\mu\partial_\mu$ in the global $U(2)$ coordinate chart angular parametrization.

21.1 Kerr-Schild ansatz

The considered above massless flat LCR-tetrad and structure relations are

$$
\begin{array}{l}
L = du - ia\overline{z^1}dz^1 + iaz^1 d\overline{z^1} \quad , \quad M = dz^1 \quad , \quad N = dv \\
\\
u := \frac{z^0 + \overline{z^0}}{2} \quad , \quad v := \frac{z^{\tilde{0}} + \overline{z^{\tilde{0}}}}{2} \\
\\
dL = 2iaM \wedge \overline{M} \quad , \quad dM = 0 \quad , \quad dN = 0
\end{array}
\tag{21.16}
$$

Notice that its relative invariants are $\Phi_1 = 2a$ and $\Phi_2 = \Phi_3 = 0$.

The LCR-tetrad implied by the Kerr-Schild ansatz is

$$
\ell = L \quad , \quad m = M \quad , \quad n = N + f(v)L
\tag{21.17}
$$

where $f(v)$ is an arbitrary function of v. The LCR-structure relations are

$$
d\ell = 2iam \wedge \overline{m} \quad , \quad dm = 0 \quad , \quad dn = -f'\ell \wedge n + 2iafm \wedge \overline{m}
\tag{21.18}
$$

Notice that the curved LCR-structure changes category with the relative invariants being now $\Phi_1 = 2a$, $\Phi_2 = 2af$ and $\Phi_3 = 0$. Notice that this is the category of the "natural $U(2)$" LCR-structure (18.14). We see that the right part of the massless soliton is no longer trivial, like in the flatprint case. Does it mean that the neutrinos small masses are due to their gravitational dressings? Or that the massless neutrino is the asymptotic plane surface of the neutrino developable LCR-surface?

21.2 Massless trajectories

The massive character of the electron in PCFT comes from the $\dot{\xi}^a \eta_{ab} \dot{\xi}^b \neq 0$ condition of its algebraic trajectory in the complex grassmannian manifold $G(4,2)$. We will now consider the simple cases of linear trajectories with null velocity. The simplest case is $\xi^a(\tau) = (\tau, 0, 0, \tau)^\top$. Then, the compatibility condition

$$\begin{pmatrix} r^0 - r^3 & -(r^1 - ir^2) \\ -(r^1 + ir^2) & r^0 + r^3 - 2\tau \end{pmatrix} \begin{pmatrix} \lambda^0 \\ \lambda^1 \end{pmatrix} = 0 \tag{21.19}$$

implies only one solution

$$\tau = \tfrac{r^a r^b \eta_{ab}}{2(r^0 - r^3)} \quad , \quad \lambda^A \sim \begin{pmatrix} r^1 - ir^2 \\ r^0 - r^3 \end{pmatrix} \tag{21.20}$$

This means that one null linear trajectory cannot determine the line of $CP(3)$, which corresponds to the point r of $G(4,2)$. Therefore we need two independent linear trajectories, one for each column of the homogeneous coordinates X^{nj} (i.e. a reducible quadric), in order to define a massless LCR-structure.

Let us consider the case of two axially symmetric massless trajectories $\xi_1^a(\tau) = (\tau, 0, 0, c_1\tau + ib_1)^\top$ and $\xi_2^a(\tau) = (\tau, 0, 0, c_2\tau + ib_2)^\top$ with $c_j = \pm 1$. Then, the explicit parametrization

$$X = \begin{pmatrix} \lambda^1 & \lambda^2 \\ -i\xi_{1a}(\tau_1)\tau^a\lambda^1 & -i\xi_{2a}(\tau_2)\tau^a\lambda^2 \end{pmatrix} \tag{21.21}$$

implies that they correspond to the following reducible quadrics

$$\begin{array}{l} For\ c_1 = 1 = c_2 \Longrightarrow \\ (iX^{21} + ib_1X^{01})(iX^{22} + ib_2X^{02}) = 0 \\ \\ For\ c_1 = 1 = -c_2 \Longrightarrow \\ (iX^{21} + ib_1X^{01})(iX^{32} - ib_2X^{12}) = 0 \\ \\ For\ c_1 = -1 = c_2 \Longrightarrow \\ (iX^{31} - ib_2X^{11})(iX^{32} - ib_2X^{12}) = 0 \end{array} \tag{21.22}$$

and the corresponding LCR-structure coordinates are

$$\begin{array}{l} For\ c_1 = 1 = c_2 \Longrightarrow\ \tau_j = \frac{1}{2}(iX^{31} - ib_jX^{11}) \\ \\ For\ c_1 = 1 = -c_2 \Longrightarrow\ \tau_1 = \frac{1}{2}(iX^{31} - ib_1X^{11}), \\ \qquad \tau_2 = \frac{1}{2}(iX^{21} + ib_2X^{01}) \\ \\ For\ c_1 = -1 = c_2 \Longrightarrow\ \tau_j = \frac{1}{2}(iX^{21} + ib_jX^{01}) \end{array} \tag{21.23}$$

In the case $c_1 = 1 = -c_2$, the compatibility conditions

$$
\begin{pmatrix} r^0 - r^3 + ib_1 & -(r^1 - ir^2) \\ -(r^1 + ir^2) & r^0 + r^3 - 2\tau_1 - ib_1 \end{pmatrix} \begin{pmatrix} \lambda^{01} \\ \lambda^{11} \end{pmatrix} = 0
$$

$$
\begin{pmatrix} r^0 - r^3 - 2\tau_2 + ib_2) & -(r^1 - ir^2) \\ -(r^1 + ir^2) & r^0 + r^3 - ib_2 \end{pmatrix} \begin{pmatrix} \lambda^{02} \\ \lambda^{12} \end{pmatrix} = 0 \tag{21.24}
$$

imply

$$
\begin{array}{l}
\tau_1 = \frac{(r^0)^2 - (r^1)^2 - (r^2)^2 - (r^3 - ib_1)^2}{2(r^0 - r^3 + ib_1)} \quad , \quad \tau_2 = \frac{(r^0)^2 - (r^1)^2 - (r^2)^2 - (r^3 - ib_2)^2}{2(r^0 + r^3 - ib_2)} \\
\lambda^{A1} \sim \begin{pmatrix} r^1 - ir^2 \\ r^0 - r^3 + ib_1 \end{pmatrix} \sim \begin{pmatrix} \frac{(r^1)^2 + (r^2)^2}{r^0 - r^3 + ib_1} \\ r^1 + ir^2 \end{pmatrix} \\
\lambda^{A2} \sim \begin{pmatrix} r^0 - r^3 + ib_2 \\ r^1 + ir^2 \end{pmatrix} \sim \begin{pmatrix} r^1 - ir^2 \\ \frac{(r^1)^2 + (r^2)^2}{r^0 + r^3 - ib_2} \end{pmatrix}
\end{array} \tag{21.25}
$$

We saw that an accelerating electron may be viewed as a ruled surface of $CP(3)$ with a non-linear trajectory $\xi^a(\tau)$ i.e.

$$
X^m = \begin{pmatrix} 1 \\ 0 \\ -i[(\xi^0 - \xi^3)] \\ -i[-(\xi^1 + i\xi^2)] \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -i[-(\xi^1 - i\xi^2)] \\ -i[(\xi^0 + \xi^3)] \end{pmatrix} \tag{21.26}
$$

Its corresponding "accelerating" neutrino is the tangential ruled (developable) surface. In order to describe this leptonic massive-massless pair we start from

$$
W^m(\tau, s) = Z^m(\tau) + sT^m(\tau) \tag{21.27}
$$

where $T^m(\tau)$ indicates the direction of the generating line which meets $Z^m(\tau)$ (the generatrix) at τ. Then in the $X^0 = 1$ chart we have

$$
\begin{pmatrix} 1 \\ \frac{Z^1 + sT^1}{Z^0 + sT^0} \\ \frac{Z^2 + sT^2}{Z^0 + sT^0} \\ \frac{Z^3 + sT^3}{Z^0 + sT^0} \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda \\ -i(\xi_{0'0} + \xi_{0'1}\lambda) \\ -i(\xi_{1'0} + \xi_{1'1}\lambda) \end{pmatrix} \tag{21.28}
$$

which implies

$$
\begin{array}{l}
\lambda = \frac{Z^1 + sT^1}{Z^0 + sT^0} \quad \Longleftrightarrow \quad s = -\frac{Z^1 - \lambda Z^0}{T^1 - \lambda T^0} \\
\\
\xi_{0'0} = i\frac{Z^2T^1 - Z^1T^2}{Z^0T^1 - Z^1T^0} \quad , \quad \xi_{0'1} = i\frac{Z^0T^2 - Z^2T^0}{Z^0T^1 - Z^1T^0} \\
\xi_{1'0} = i\frac{Z^3T^1 - Z^1T^3}{Z^0T^1 - Z^1T^0} \quad , \quad \xi_{1'1} = i\frac{Z^0T^3 - Z^3T^0}{Z^0T^1 - Z^1T^0} \\
Z^0T^1 - Z^1T^0 \neq 0
\end{array} \tag{21.29}
$$

for the accelerating electron. For the "accelerating" electronic neutrino ($T^n = \dot{Z}$) we find

$$
\begin{array}{ll}
\lambda = \frac{Z^1+s\dot{Z}^1}{Z^0+s\dot{Z}^0} \quad \Longleftrightarrow \quad s = -\frac{Z^1-\lambda Z^0}{\dot{Z}^1-\lambda\dot{Z}^0} & \\
\xi_{0'0} = i\frac{Z^2\dot{Z}^1-Z^1\dot{Z}^2}{Z^0\dot{Z}^1-Z^1\dot{Z}^0} \quad , \quad \xi_{0'1} = i\frac{Z^0\dot{Z}^2-Z^2\dot{Z}^0}{Z^0\dot{Z}^1-Z^1\dot{Z}^0} & \\
\xi_{1'0} = i\frac{Z^3\dot{Z}^1-Z^1\dot{Z}^3}{Z^0\dot{Z}^1-Z^1\dot{Z}^0} \quad , \quad \xi_{1'1} = i\frac{Z^0\dot{Z}^3-Z^3\dot{Z}^0}{Z^0\dot{Z}^1-Z^1\dot{Z}^0} & \\
Z^0\dot{Z}^1 - Z^1\dot{Z}^0 \neq 0 &
\end{array}
\tag{21.30}
$$

Notice that the vanishing of the Gaussian curvature condition

$$
\dot{\xi}_{0'0}\dot{\xi}_{1'1} - \dot{\xi}_{0'1}\dot{\xi}_{1'0} = 0 \tag{21.31}
$$

for the neutrino is satisfied, as expected. Besides in the $Z^0 = 1$ patch, the developable surface takes the form

$$
\begin{array}{ll}
\lambda = Z^1 + s\dot{Z}^1 \quad \Longleftrightarrow \quad s = -\frac{Z^1-\lambda}{\dot{Z}^1} & \\
\xi_{0'0} = i\frac{Z^2\dot{Z}^1-Z^1\dot{Z}^2}{\dot{Z}^1} \quad , \quad \xi_{0'1} = i\frac{\dot{Z}^2}{\dot{Z}^1} & \\
\xi_{1'0} = i\frac{Z^3\dot{Z}^1-Z^1\dot{Z}^3}{\dot{Z}^1} \quad , \quad \xi_{1'1} = i\frac{\dot{Z}^3}{\dot{Z}^1} & \\
\dot{Z}^1 \neq 0 &
\end{array}
\tag{21.32}
$$

As a developable surface $W^n(\tau, s)$ of $CP(3)$, the neutrino needs some better understanding. The parametrization

$$
dW^m(\tau, s) = (\dot{Z}^m + s\ddot{Z}^m)d\tau + \dot{Z}^m ds \tag{21.33}
$$

implies that $\dot{Z}^m$ and $\ddot{Z}^m$ must be linearly independent and $s \neq 0$. Hence the Darboux frame coincides with the Frenet arclength frame of the holomorphic curve $Z^m(\tau)$. In the $Z^0 = 1$ euclidian metric we find that the Cartan lift of the holomorphic curve $Z^i(\tau)$ is (13.46).

22 GRAVITATIONAL AND ELECTROWEAK CONNECTIONS

In conventional field theory the interactions have to be imposed as connections. In the computation of the electron and its neutrino distributional LCR-structures, these fields appear as gravitational and electromagnetic dressing distributions with precise compact singular support. The purpose of the present section is to provide the algorithmic derivation of the Einstein and the weak connections. The surprising result is that both connections are manifestations of the same quantities, the LCR-tetrad. This essentially generalizes the surprising identification of the electron electromagnetic dressing $A_\mu(x)$ with the vector ℓ_μ of the LCR-tetrad (and the induced gravitational dressing).

The definition of the LCR-tetrad is

$$\begin{aligned}
d\ell &= Z_1 \wedge \ell + i\Phi_1 m \wedge \overline{m} \\
dm &= Z_3 \wedge m + \Phi_3 \ell \wedge n \\
\\
dn &= Z_2 \wedge n + i\Phi_2 m \wedge \overline{m} \\
d\overline{m} &= \overline{Z_3} \wedge \overline{m} + \overline{\Phi_3} \ell \wedge n
\end{aligned} \tag{22.1}$$

where Z_1, Z_2 are real 1-forms, Z_3 a complex 1-form, Φ_1, Φ_2 two real scalars and Φ_3 a complex scalar. Apparently, it is not invariant under the usual $SO(1,3)$ transformation of ordinary Einstein connection of general relativity. If we fix the tetrad-Weyl symmetry, we may osculate the LCR-tetrad with the $SO(1,3)$ connection. The existence of the LCR-tetrad may be viewed as a breaking of the $SO(1,3)$ symmetry down to the possible LCR-tetrad.

The general solution of a realizable LCR-structure is a special totally real submanifold of $\mathbb{C}^4$, determined by the following conditions

$$\begin{aligned}
&\rho_{11}(\overline{z^\alpha}, z^\alpha) = 0 \quad , \quad \rho_{12}\left(\overline{z^\alpha}, z^{\widetilde{\alpha}}\right) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\alpha}}) = 0 \\
&\tfrac{\partial \rho_{ij}}{\partial z^b} \neq 0 \neq \tfrac{\partial \rho_{ij}}{\partial \overline{z^b}}
\end{aligned} \tag{22.2}$$

Its characteristic local coframe of the surface contains the normal bundle $d\rho_{ij}$ and the tangent 1-forms

$$\begin{aligned}
&\ell = i(\partial - \overline{\partial})\rho_{11} \quad , \quad n = i(\partial - \overline{\partial})\rho_{22} \quad , \quad m = i(\partial - \overline{\partial})\overline{\rho_{12}} \\
&\begin{pmatrix} \ell & \overline{m} \\ m & n \end{pmatrix} = i(\partial - \overline{\partial}) \begin{pmatrix} \rho_{11} & \rho_{12} \\ \overline{\rho_{12}} & \rho_{22} \end{pmatrix}
\end{aligned} \tag{22.3}$$

The ordinary theory of surfaces in general position does not apply here, because the ambient Kaehler manifold does not have a precise metric and we must generally permit the existence of distributional singularities in order to incorporate the distributional solitons. Therefore we have to consider connections with sources in the context of representations of Poincaré group.

We already know that the above tangent vectors (the LCR-tetrad) determine a Hodge structure, which played an essential role to the determination of the electron and its neutrino. The Einstein metric is an element of the class (relative to the tetrad-Weyl) of symmetric tensors $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, where e^a_μ is a general null tetrad (but not always geodetic and shear-free) and η_{ab} is the well known form of the $SO(1,3)$ invariant Minkowski metric adapted to the null vectors. Apparently a $SO(1,3)$ transformation breaks the tetrad-Weyl symmetry and vice-versa, the tetrad-Weyl transformation breaks the $SO(1,3)$ connection. Besides, an LCR-tetrad implies only Einstein metrics, which admit a geodetic and shear-free null tetrad, which turns out to be principal directions of the corresponding conformal curvature. A non singular ($\det g_{\mu\nu} \neq 0$) symmetric tensor defines a $SO(1,3)$ riemannian structure. It is well known that the corresponding Cartan connection is implied by the existence of the internal product

$$\begin{aligned}
&(e^a, e^b) := e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab} \quad \rightarrow \quad (de^a, e^b) + (e^a, de^b) = 0 \\
&de^a = \omega^a{}_c e^c \quad \rightarrow \quad \omega^a{}_c \eta^{cb} + \eta^{ad} \omega^b{}_d = 0
\end{aligned} \tag{22.4}$$

which is the $SO(1,3)$ Cartan connection. Because of its $SO(1,3)$ covariance, it is essentially computed by the LCR-tetrad despite the fact that its transformed null tetrads e^a_μ are **not** LCR-tetrads.

We will now prove that the above definition of the Einstein connection permits the emergence of distributional singularities in the embedded LCR-manifold. The starting point is to write the surface $\rho_{ij} = 0$, using the regular coordinates

$$\begin{aligned}
&\operatorname{Im} z^0 = \phi_{11}(\overline{z^1}, z^1, \operatorname{Re} z^0) \ , \ \operatorname{Im} z^{\widetilde{0}} = \phi_{22}(\overline{z^{\widetilde{1}}}, z^{\widetilde{1}}, \operatorname{Re} z^{\widetilde{0}}) \ , \ z^{\widetilde{1}} - \overline{z^1} = \phi_{12}(\overline{z^\beta}, z^{\widetilde{0}}) \\
&\phi_{11}(p) = \phi_{22}(p) = \phi_{12}(p) = 0 \quad , \quad d\phi_{11}(p) = d\phi_{22}(p) = d\phi_{12}(p) = 0
\end{aligned} \tag{22.5}$$

in a neighborhood of a point p. But the LCR-structure is a special totally real CR-structure, which at a real analytic neighborhood admits a general analytic transformation $r^b = f^b(z^c)$, which makes it trivial

$$\tfrac{r^a - \overline{r^a}}{2i} = 0 \tag{22.6}$$

This last analytic transformation is not generally an LCR-transformation. Hence, it breaks the LCR-structure, but there is no reason to worry for that now, because we are going to look for a connection, which has already broken the LCR-symmetry. The essential point here is the neighborhood of p in the ambient complex manifold, where the analytic transformation can be extended. The case of the distributional electron (and neutrino) indicates that the analytic transformation cannot extend around their location. Besides the entire region of the LCR-manifold can be described by a distribution with a representative (locally integrable function), which at the regular point p appears as a regular potential with each source at the location of the electron. Besides, the location of the electron is not a real analytic region of the LCR-manifold, because it does not admit analytic extensions in both sides in the ambient complex manifold. Recall that it is the Sato's definition of generalized functions. In the neighborhood of the regular point p the Kaehler potential becomes $(r^a - \overline{r^a})(r^b - \overline{r^b})\eta_{ab}$ and the induced metric is the Minkowski metric. Hence at this point, the gravitational dressing is regularly expanded around the Minkowski metric.

Recall that the "natural U(2)" LCR-structure (4.33) is

$$\begin{aligned}
&e = -iw^{-1}dw =: \begin{pmatrix} \ell & \overline{m} \\ m & n \end{pmatrix} \quad , \quad de - ie \wedge e = 0 \\
&d\ell = im \wedge \overline{m} \quad , \quad dn = -im \wedge \overline{m} \quad , \quad dm = i(\ell - n) \wedge m
\end{aligned} \tag{22.7}$$

This form strongly suggests to osculate the LCR-structure with the $U(2)$ group. The first step of that is to cast a LCR-tetrad into the hermitian matrix

$$e' := \begin{pmatrix} \ell' & \overline{m'} \\ m' & n' \end{pmatrix} = i(\partial - \overline{\partial}) \begin{pmatrix} \rho_{11} & \rho_{12} \\ \overline{\rho_{12}} & \rho_{22} \end{pmatrix} \tag{22.8}$$

Following the Maurer-Cartan procedure we consider the hermitian matrix e an elements of $u(2)$ Lie algebra, i.e. a $U(2)$ connection with non-vanishing curvature. The connection and the corresponding curvature

$$\begin{gathered} B = B_{I\mu} dx^\mu t_I = \begin{pmatrix} \ell' & \overline{m'} \\ m' & n' \end{pmatrix} \quad , \quad [t_I, t_J] = iC_{IJK} t_K \\ F = dB - iB \wedge B \ \longrightarrow DF := \ dF + iB \wedge F - iF \wedge B = 0 \end{gathered} \tag{22.9}$$

where t_J are generators of $U(2)$. Apparently a gauge transformation breaks the tetrad-Weyl symmetry. That is, the $U(2)$ transformation is expected to transform LCR-structures to other LCR-structures, like the weak $U(2)$ transforms electron to its neutrino and vice-versa. Therefore we chose the LCR-tetrad e' is such that $\Phi'_1 = 1 = -\Phi'_2$. That is, we partly fix the tetrad-Weyl symmetry for non-trivial LCR-structures with $\Phi_1 \neq 0 \neq \Phi_2$. Recall the general tetrad-Weyl transformation

$$\begin{array}{l} \ell' = \Lambda \ell \quad , \quad n' = N n \quad , \quad m' = M m \\ \\ Z'_1 = Z_1 + d(\ln \Lambda) \quad , \quad \Phi'_1 = \frac{\Lambda}{M\overline{M}} \Phi_1 \\ Z'_2 = Z_2 + d(\ln N) \quad , \quad \Phi'_2 = \frac{N}{M\overline{M}} \Phi_2 \\ Z'_3 = Z_3 + d(\ln M) \quad , \quad \Phi'_3 = \frac{M}{\Lambda N} \Phi_3 \end{array} \tag{22.10}$$

Like the $O(1,3)$ connection, the present $U(2)$ group does not preserve the LCR-structure conditions. In the case of the following generators

$$t_0 = I \quad , \quad t_k = \tfrac{\sigma_k}{2} \ \rightarrow \ C_{ijk} = \epsilon_{ijk} \tag{22.11}$$

we have

$$\begin{array}{l} B_{0\mu} + \frac{1}{2} B_{3\mu} = \ell'_\mu \quad , \quad B_{0\mu} - \frac{1}{2} B_{3\mu} = n'_\mu \quad , \quad \frac{1}{2}(B_{1\mu} + iB_{2\mu}) = m'_\mu \\ \\ F_{0\mu\nu} = \partial_\mu B_{0\nu} - \partial_\nu B_{0\mu} \\ F_{i\mu\nu} = \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu} - \epsilon_{ijk} B_{j\mu} B_{k\nu} \end{array} \tag{22.12}$$

The standard model model relations between the $U(1)$ gauge potential $B_{0\mu}$ and the $SU(2)$ gauge potentials $B_{j\mu}$ suggest us to identify the electromagnetic potential A_μ with ℓ'_μ, the neutral potential Z_μ with n'_μ and the charged potential W_μ with m'_μ. Besides, the relative invariants are apparently related to the Higgs field.

Hence, the Einstein goal to unify all the potentials (gravitation, electroweak and Higgs) seems to be achieved by the consideration of the LCR-structure as the fundamental geometric structure. Besides, the identification of the electromagnetic potential with a multiple of the null tetrad ℓ_μ in the Kerr-Newman solution is not a computational accident!

22.1 Electroweak and Higgs potentials of the electron

In the case of the electron LCR-tetrad (19.9) the three relative invariants are (19.18)

$$\begin{array}{l} \Phi_1 = \frac{-2a\cos\theta}{r^2+a^2\cos^2\theta} \\ \Phi_2 = -\frac{(r^2+a^2+h)a\cos\theta}{(r^2+a^2\cos^2\theta)^2} \\ \Phi_3 = \frac{2iar\sin\theta}{\sqrt{2}(r+ia\cos\theta)^2(r-ia\cos\theta)} \end{array} \tag{22.13}$$

We first make the tetrad-Weyl transformation to reach the conditions $\Phi'_1 = 1 = -\Phi'_2$. We find

$$\begin{array}{l} N = -\frac{r^2+a^2\cos^2\theta}{r^2+a^2+h}\Lambda \\ M\overline{M} = -\frac{2a\cos\theta}{r^2+a^2\cos^2\theta}\Lambda \end{array} \tag{22.14}$$

The electromagnetic dressing (19.78) is found with $\Lambda = \frac{qr}{r^2+a^2\cos^2\theta}$. Then the electroweak connection B (22.12) is found with

$$\begin{array}{l} \Lambda = \frac{qr}{r^2+a^2\cos^2\theta} \\ N = -\frac{qr}{4\pi(r^2+a^2+h)} \\ M\overline{M} = -\frac{qra\cos\theta}{2\pi(r^2+a^2\cos^2\theta)^2} \end{array} \tag{22.15}$$

up to an M phase tetrad-Weyl transformation.That is, we find the following electroweak potentials (dressings) of the electron

$$\begin{array}{l} A = \frac{qr}{4\pi(r^2+a^2\cos^2\theta)}(dt - dr - a\sin^2\theta d\varphi) \\ Z = \frac{-qr}{8\pi(r^2+a^2\cos^2\theta)}(dt + \frac{r^2+2a^2\cos^2\theta-a^2-h}{r^2+a^2+h}dr - a\sin^2\theta d\varphi) \\ W = \frac{-M}{\sqrt{2}(r+ia\cos\theta)}[-ia\sin\theta\ (dt-dr) + (r^2+a^2\cos^2\theta)d\theta + \\ \qquad +i\sin\theta(r^2+a^2)d\varphi] \end{array} \tag{22.16}$$

where the tetrad-Weyl factor M will be computed below through the Higgs dressing.

The third (complex) relative invariant Φ'_3 (22.10) is not completely fixed. Its phase is absorbed by W and the remaining scalar real field Φ'_3 will be finally related with the electron Higgs potential

$$\begin{array}{l} M = \frac{i}{r-ia\cos\theta}\big[\frac{qra\cos\theta}{2\pi(r^2+a^2\cos^2\theta)^2}\big]^{\frac{1}{2}} \\ \Phi'_3 = \frac{2\sin\theta(r^2+a^2+h)}{r^2+a^2\cos^2\theta}\big[\frac{\pi a^3\cos\theta}{q^3 r}\big]^{\frac{1}{2}} \end{array} \tag{22.17}$$

We see that the Einstein's gravitational connection and the $U(2)$ gauge field are directly related to the LCR-tetrad and the Higgs field is related with the Φ_i relative invariants of the LCR-structure. In Part IV (On the "Origin" of Quantum Mechanics) of this Research eBook, we will further exploit these relations with perturbative standard model and quantum gravity calculations in the context of Bogoliubov recursive causal approach.

The tetrad-Weyl transformation (22.10) is the local symmetry of the fundamental geometric LCR-structure. This symmetry is broken by the local Lorentz $SO(1,3)$ transformation, which preserves the Einstein metric. It is also broken by the the electroweak $U(2)$ transformation. That is the $SO(1,3)$ and

$U(2)$ transformations are transversal to the tetrad-Weyl transformations. The tetrad-Weyl factors (22.15) are fixed by assuming the appropriate Λ factor that provides the charge conserving electromagnetic field.

23 MUON AND TAU GENERATIONS

Quantum standard model of electroweak interactions is actually the most general and precise model in quantum field theory. It is based on the correspondence of each "elementary particle" with a quantum field representation of the Poincaré group. It does not explain how the trajectory of the elementary particle emerges. The trajectory can enter quantum field theory through a solitonic configuration of the particle. In PCFT the electron is a precise solitonic LCR-manifold, which satisfies charge conservation law. In the zero gravity approximation, they are LCR-structures of the boundary of the $SU(2,2)$ symmetric classical domain. The study of the electron LCR-structure revealed the importance of the ruled surfaces of $CP(3)$, which are related to complex trajectories in the grassmannian $G(4,2)$. These are surfaces generated by lines joining points of two holomorphic curves $Z_i^n(\tau),\ i=1,2$ in $CP(3)$, corresponding to the same variable τ. That is, the general explicit form of a ruled surface is

$$\begin{aligned} Z^m(\tau,s) = (1-s)Z_1^m(\tau) + sZ_2^m(\tau) = \\ = Z_1^m(\tau) + s[Z_2^m(\tau) - Z_1^m(\tau)] \end{aligned} \tag{23.1}$$

The generating lines (rulings) correspond to complex points of the grassmannian manifold $G(4,2)$, with projective coordinates

$$\begin{aligned} & \xi(\tau) =: iX_2X_1^{-1} =: \begin{pmatrix} \xi^0 - \xi^3 & -(\xi^1 - i\xi^2) \\ -(\xi^1 + i\xi^2) & \xi^0 + \xi^3 \end{pmatrix} \\ & X_1 =: \begin{pmatrix} Z_1^0 & Z_2^0 \\ Z_1^1 & Z_2^1 \end{pmatrix} \quad , \quad X_2 =: \begin{pmatrix} Z_1^2 & Z_2^2 \\ Z_1^3 & Z_2^3 \end{pmatrix} \end{aligned} \tag{23.2}$$

Using homogeneous coordinates, this curve of $G(4,2)$ is spanned by the two vectors $Z_i^n(\tau);\ i=1,2$. The curve is called non-degenerate if the following determinant does not identically vanish

$$\begin{aligned} & \det[Z_1^n, Z_2^n, \tfrac{dZ_1^n}{d\tau}, \tfrac{dZ_2^n}{d\tau}] = \det \begin{pmatrix} X_1 & \dot{X}_1 \\ -i\xi X_1 & -i(\dot{\xi}X_1 + \xi\dot{X}_1) \end{pmatrix} = \\ & = \det[\begin{pmatrix} 1 & 0 \\ -i\xi & 1 \end{pmatrix} \begin{pmatrix} X_1 & \dot{X}_1 \\ 0 & -i\dot{\xi}X_1 \end{pmatrix}] = -\det(\dot{\xi})(\det X_1)^2 \end{aligned} \tag{23.3}$$

This happens if and only if $\dot{\xi}^a \dot{\xi}^b \eta_{ab} \neq 0$. If it vanishes, the Gauss curvature of the surface vanishes and the ruled surface is called developable. That is we have a pair of massive and massless LCR-structures characterized by a generally complex trajectory. The precise forms of the electron and its neutrino has been explicitly derived. Let me remind you their general formulation.

In the coordinate chart $X^0 = 1$, a general point of the ruled surface determined by a trajectory $\xi^b(\tau)$ has the form

$$
\begin{array}{l}
X^n(\tau,\lambda) = \begin{pmatrix} 1 \\ \lambda \\ -i[(\xi^0-\xi^3)-(\xi^1-i\xi^2)\lambda] \\ -i[-(\xi^1+i\xi^2)+(\xi^0+\xi^3)\lambda] \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -i(\xi^0-\xi^3) \\ i(\xi^1+i\xi^2) \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ i(\xi^1-i\xi^2) \\ -i(\xi^0+\xi^3) \end{pmatrix} \\
\lambda =: (1-s)Z_1^1(\tau) + sZ_2^1(\tau)
\end{array}
\tag{23.4}
$$

The first term is the "directrix curve" of the ruled surface and the second is the generating line ("ruling") of the surface. This is the form we have already assumed in order to introduce the trajectory of the electron. And precisely the linear trajectory $\xi^b(\tau) = v^b\tau + d^b$ with $(v^b)^2 = 1$ corresponds to the "free" electron and with $(v^b)^2 = 0$ corresponds to its neutrino.

Two independent points $X^{ni} = X^n(\tau_i, \lambda_i)$ of a ruled surface determine a line of $CP(3)$, which **does not belong** to the ruled surface and hence its corresponding point in $G(4,2)$ does not belong to the complex trajectory of the ruled surface. The two general points $X^{ni} \in CP(3)$, viewed as a point of the grassmannian $G(4,2)$, satisfy the identity

$$
\begin{array}{l}
X^{mj} = \begin{pmatrix} \lambda^{Aj} \\ -ir_{A'B}\lambda^{Bj} \end{pmatrix} = \begin{pmatrix} \lambda^{Aj} \\ -i\xi_{A'B}\lambda^{Bj} \end{pmatrix} \\
\lambda^{Aj} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}
\end{array}
\tag{23.5}
$$

where the 2×2 matrix $r_{A'B}$ are its projective coordinates. In the case of zero gravity, it becomes hermitian and it is denoted $x_{A'B}$, being a point of the boundary of the $SU(2,2)$ classical domain. Then we find the two roots (τ_i, λ_i) of the matrix relation

$$
[x_{A'B} - \xi_{A'B}(\tau_j)]\lambda^{Bj} \tag{23.6}
$$

Recall that in the electron LCR-structure, the two roots of (τ_i, λ_i) are assumed as the natural LCR-structure coordinates $(z^\alpha, z^{\widetilde{\beta}})$, which satisfy the relations

$$
\begin{array}{lll}
z^0 := \tau_1(x) & , & z^1 = \frac{(x^1+ix^2)-(\xi^1(z^0)+i\xi^2(z^0))}{(x^0+x^3)-(\xi^0(z^0)+\xi^3(z^0))} \\
z^{\widetilde{0}} := \tau_2(x) & , & z^{\widetilde{1}} = -\frac{(x^1-ix^2)-(\xi^1(z^{\widetilde{0}})-i\xi^2(z^{\widetilde{0}}))}{(x^0-x^3)-(\xi^0(z^{\widetilde{0}})-\xi^3(z^{\widetilde{0}}))}
\end{array}
\tag{23.7}
$$

and the LCR-tetrad (in the zero gravity approximation) becomes

$$
\begin{array}{l}
L = \frac{1}{\sqrt{2}}\bar{o}^{A'1}\, dx_{A'A}o^{A1} \quad , \quad N = \frac{1}{\sqrt{2}}\bar{o}^{A'2}\, dx_{A'A}o^{A2} \quad , \quad M^\mu = \frac{1}{\sqrt{2}}\,\bar{o}^{A'2}\, dx_{A'A}o^{A1} \\
o^{Ai} =: \frac{1}{1+z^1z^{\widetilde{1}}} \begin{pmatrix} 1 & -z^{\widetilde{1}} \\ z^1 & 1 \end{pmatrix} \quad , \quad o^{A1}o^{B2}\epsilon_{AB} = 1
\end{array}
\tag{23.8}
$$

This profound origin of classical trajectory permit us to distinguish the spinorial "elementary particles", as those based on ruled surfaces, from the rest LCR-structures. It is well known that the other two leptonic families, (μ, ν_μ) and

(τ, ν_τ) are completely analogous to the electron. The muon is created at the upper atmosphere and travels to the earth surface as a (free) particle. That is, my **assertion** is that the leptonic generations (e, ν_e), (μ, ν_μ) and (τ, ν_τ) are three pairs of LCR-structures determined by ruled surfaces related to (electron-like) complex trajectories. The conserved three leptonic numbers (electronic, muonic and tauonic) are different topological Hopf invariants of the corresponding ruled LCR-structures with possible linear complex trajectory. In the following subsections I will specify these LCR-structures and I will compute the corresponding Hopf invariants. I think that this computation will make clear that in the context of PCFT the conserved three leptonic numbers are the topological Hopf invariants of the left and right chiral parts of the X^{ni} homogeneous coordinates (in the zero gravity approximation).

23.1 The Hopf invariant of the electron generation

We have already pointed out that the "natural $U(2)$" LCR-structure has an internal topological linking number (18.22), which is easily viewed in its simple bounded realization. In order to understand the electron (and neutrino) Hopf invariant we have to consider the global view of the electron and neutrino LCR-structures in the zero gravity approximation, i.e. $U(2)$ surface of $G(4,2)$. The quadratic Kerr polynomials of the electron and neutrino projectively imply two solutions $\lambda^{Ai}(x)$ $i = 1, 2$ in S^2. That is, we have two functions

$$S^1 \times S^3 \to S^2 \tag{23.9}$$

one for the left column and one for the right column of the homogeneous grassmannian coordinates. It is known that the homotopy group $\pi_1(S^2)$ is trivial but $\pi_3(S^2) = \mathbb{Z}$. The Hopf invariant of every column is determined using the sphere volume 2-form

$$\omega = \frac{i}{2\pi} \frac{d\lambda \wedge d\overline{\lambda}}{(1 + \lambda\overline{\lambda})^2} \tag{23.10}$$

which is closed. This implies that in S^3 there is an exact 1-form ω_1 such that $\omega = d\omega_1$. Then the Hopf invariant of $\lambda(x)$ is

$$H(\lambda) = \int \lambda^*(\omega) \wedge \omega_1 \tag{23.11}$$

We will also start from the computationally simple neutrino case in the two $\mathbb{R}^4$-sheets affine space

$$X^{ni} = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ X^{21} & X^{22} \\ X^{31} & X^{32} \end{pmatrix} =: \begin{pmatrix} \lambda \\ -ix\lambda \end{pmatrix} \quad , \quad \det \lambda \neq 0 \tag{23.12}$$

of the grassmannian $G(4,2)$, where the Kerr polynomials are

$$bX^{11}+X^{31}=0 \quad , \quad X^{02}=0 \tag{23.13}$$

for the left and right columns of the homogeneous coordinates. The left column solution is

$$\lambda(x)=\frac{x^1+ix^2}{x^3+ib} \tag{23.14}$$

where its restriction $x^0=0$ to $SU(2)$ is assumed. I find

$$\begin{aligned}
d\lambda &= \frac{(x^3+ib)(dx^1+idx^2)-(x^1+ix^2)dx^3}{(x^3+ib)^2} \\
\omega &= \frac{i}{2\pi}\frac{d\lambda\wedge d\overline{\lambda}}{(1+\lambda\overline{\lambda})^2} = \frac{((x^3)^2+b^2)dx^1\wedge dx^2+(bx^1-x^2x^3)dx^1\wedge dx^3+(bx^2+x^1x^3)dx^2\wedge dx^3}{\pi[(x^1)^2+(x^2)^2+(x^3)^2+b^2]^2}
\end{aligned} \tag{23.15}$$

which implies

$$\omega_1=\frac{-x^2dx^1+x^1dx^2-bdx^3}{2\pi[(x^1)^2+(x^2)^2+(x^3)^2+b^2]} \tag{23.16}$$

Then we finally find

$$H(\lambda)=\frac{-b}{|b|} \tag{23.17}$$

where I used the integral formula $\int\limits_0^\infty \frac{x^2dx}{(x^2+1)^2}=\frac{\pi}{4}$ and I have integrated over the two $\mathbb{R}^4$-sheets, which cover S^3 by simply permitting $r\in(-\infty\ ,\ +\infty)$. The consideration of the negative r sheet is essential, otherwise we could not find the expected integer value integer value for the topological number.

The Hopf invariant of the right column X^{n2} vanishes, because the CR-structure is degenerate.

Recall that the static electron solitonic LCR-manifold is given by the irreducible quadratic Kerr polynomial (in the unbounded Siegel realization)

$$X^1X^2-X^0X^3+2aX^0X^1=0 \tag{23.18}$$

of $CP(3)$. In the flatprint case we have

$$\begin{array}{l}
X^0=1 \quad , \quad X^1=\lambda \quad , \quad X^2=-i[(x^0-x^3)-(x^1-ix^2)\lambda] \\
X^3=-i[-(x^1+ix^2)+(x^0+x^3)\lambda]
\end{array} \tag{23.19}$$

and the Kerr polynomial and its two solutions are

$$\begin{array}{l}
(x^1-ix^2)\lambda^2+2(x^3-ia)\lambda-(x^1+ix^2)=0 \\
\lambda_{1,2}=\frac{-(x^3-ia)\pm\sqrt{\Delta}}{x-iy} \quad , \quad \Delta=(x^1)^2+(x^2)^2+(x^3)^2-a^2-2iax^3
\end{array} \tag{23.20}$$

where $\lambda_{1,2}$ are the two values of λ on the two sheets of the quadric. In the present case, it is convenient to compute the Hopf invariants of the left and right columns of X^{ni} using its relation to the linking coefficient of two closed curves in S^3 determined by the inverse images $\lambda^{-1}(\lambda_1)=\{x_1^i(\rho_1)\}$ and $\lambda^{-1}(\lambda_2)=\{x_2^i(\rho_2)\}$. Two general closed curves are determined using the Lindquist coordinates $(\rho,\ \theta,\ \varphi)$

$$x^i=(\sin\theta\cos\varphi\ ,\ \sin\theta\sin\varphi\ ,\ \cos\theta)\rho+a(\sin\varphi\ ,\ -\cos\varphi\ ,\ 0) \tag{23.21}$$

for two different values of θ,φ and the variable $\rho\in(-\infty,\ +\infty)$ in order to cover the whole sphere. Then we know that

$$H(\lambda)=2\frac{1}{4\pi}\int\frac{\varepsilon_{ijk}(x_1^i-x_2^i)dx_1^j dx_2^k}{|\overrightarrow{x_1}-\overrightarrow{x_2}|^3} \tag{23.22}$$

The two curves can be smoothly deformed to the values $\theta_1=0$ and $\theta_2=\frac{\pi}{2}\ ,\ \varphi_2=0$. Then the integral becomes

$$H(\lambda_\pm)=\frac{a}{2\pi}\int\int\frac{d\rho_1 d\rho_2}{(\rho_1^2+\rho_2^2+a^2)^{\frac{3}{2}}}=\pm\frac{a}{|a|} \tag{23.23}$$

Notice that the left and right Hopf invariants are the opposite electron helicities.

Concluding this subsection I want to point out that a classical weak current based on a trajectory starting with velocity $(\dot{\xi})^2=1$ and finishing with $(\dot{\xi})^2=0$ (what ever it means in the classical de Rham current formulation of the generalized functions) may preserve **only** the left Hopf invariant, because the right Hopf invariant of the neutrino vanishes. Hence we may identify the conserved electronic leptonic number with the Hopf invariant! In the next subsection I will generalize this identification to the other two leptonic numbers (muonic and tauonic) and I will compute them.

23.2 Linking numbers of the three leptonic generations

The complete similarities of the muon and tau heavy leptons with the electron suggests that their LCR-structures to be based on ruled surfaces of $CP(3)$ with the linear complex trajectory of the free electron, but with different $z^1(x)$ and $z^{\widetilde{1}}(x)$ coordinates which have higher Hopf numbers. I will now show that it is possible.

Recall that the static complex trajectory $\xi^a(\tau)=(\tau,0,0,ia)$ has the homogeneous coordinates

$$X^{ni}=\begin{pmatrix}X^{01} & X^{02}\\ X^{11} & X^{12}\\ X^{21} & X^{22}\\ X^{31} & X^{32}\end{pmatrix}=\begin{pmatrix}1 & -z^{\widetilde{1}}\\ z^1 & 1\\ -i(z^0-ia) & i(z^{\widetilde{0}}-ia)z^{\widetilde{1}}\\ -i(z^0+ia)z^1 & -i(z^{\widetilde{0}}+ia)\end{pmatrix} \tag{23.24}$$

which satisfy the (zero gravity) LCR-structure conditions

$$
\begin{gathered}
X^{ni}X^{mj}E^{U}_{nm}=0 \quad , \quad E^{U}=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\
\tfrac{z^0-\overline{z^0}}{2i}-a\tfrac{1-z^1\overline{z^1}}{1+z^1\overline{\overline{z^1}}} \quad , \quad \tfrac{z^{\tilde{0}}-\overline{z^{\tilde{0}}}}{2i}+a\tfrac{1-z^{\tilde{1}}\overline{z^{\tilde{1}}}}{1+z^{\tilde{1}}\overline{\overline{z^{\tilde{1}}}}} \quad , \quad z^{\tilde{1}}-\overline{z^1}\tfrac{\overline{z^0}-z^{\tilde{0}}-2ia}{\overline{\overline{z^0}}-z^{\tilde{0}}+2ia}=0
\end{gathered}
\tag{23.25}
$$

The structure coordinates $z^b(x)$ are determined by

$$
(x_{A'B}-\xi_{A'B}(\tau))\lambda^{Bj}=0 \tag{23.26}
$$

That is $z^0(x)$ and $z^{\tilde{0}}(x)$ are the two roots of

$$
\begin{gathered}
\det(x_{A'B}-\xi_{A'B}(\tau_i))=0 \\
z^0(x)=\tau_1 \quad , \quad z^{\tilde{0}}(x)=\tau_2
\end{gathered}
\tag{23.27}
$$

Hence they are completely determined by the trajectory $\xi^b(\tau)$ of the ruled surface and are the same for the corresponding massive and massless particles of the three different particles of the leptonic generations. On the other hand the left $z^1(x)$ and right $z^{\tilde{1}}(x)$ structure coordinates are

$$
z^1(x)=\tfrac{\lambda^{11}}{\lambda^{01}} \quad , \quad z^{\tilde{1}}(x)=-\tfrac{\lambda^{02}}{\lambda^{12}} \tag{23.28}
$$

with higher Hopf numbers.

Once we have identified the electron with the left and right Hopf numbers $(+1,-1)$ and its neutrino with left and right Hopf numbers $(+1,0)$ the expected higher Hopf numbers $(+k,-k)$ and $(+k,0)$ are usually implied by composition with higher k-degree mappings $f: S^2 \to S^2$ between the 2-spheres. That is we take $f(\tan\frac{\theta}{2}e^{i\varphi})=\tan\frac{\theta}{2}e^{ik\varphi}$.

In the flatprint electron LCR-structure, the relation between the cartesian coordinates and the structure coordinates are

$$
\begin{aligned}
&x^0=t \\
&x^1+ix^2=(r-ia)\sin\theta e^{i\varphi} \\
&x^3=r\cos\theta
\end{aligned}
\tag{23.29}
$$

For constant time, the (left) causal ray $\ell^\mu(r)$ is

$$
\begin{aligned}
&x^0=t=0 \\
&x^1=(r\cos\varphi+a\sin\varphi)\sin\theta \\
&x^2=(r\sin\varphi-a\cos\varphi)\sin\theta \\
&x^3=r\cos\theta
\end{aligned}
\tag{23.30}
$$

Recall that the entire $SU(2)$ space is covered by considering $r\in(-\infty,+\infty)$. This 3-dimensional space may be considered as the initial data of the electron LCR-manifold.

Let us now consider the following initial data

$$\begin{array}{l} x^0 = t = 0 \\ x^1 = (r\cos k\varphi + a\sin k\varphi)\sin\theta \\ x^2 = (r\sin k\varphi - a\cos k\varphi)\sin\theta \\ x^3 = r\cos\theta \\ \\ r \in (-\infty, +\infty), \quad \theta \in (0,\pi), \quad \varphi \in (0, 2\pi), \quad k \in \mathbb{Z} \end{array} \tag{23.31}$$

and compute the linking number of the circles

$$\begin{array}{l} \overrightarrow{x'} = (0,\ 0,\ r) \quad , \quad \theta = 0,\ \varphi = 0 \\ \overrightarrow{x} = (a\sin k\varphi,\ -a\cos k\varphi,\ 0) \quad , \quad r = 0,\ \theta = \frac{\pi}{2} \\ \\ l = \frac{1}{4\pi}\iint \frac{\epsilon_{ijk}(x'^i - x^i)dx^j \wedge dx'^k}{[\sum\limits_i (x'^i - x^i)^2]^{\frac{3}{2}}} \end{array} \tag{23.32}$$

I find

$$\begin{array}{l} \overrightarrow{x'} = (0,\ 0,\ r) \quad , \quad \theta = 0,\ \varphi = 0 \\ \overrightarrow{x} = (a\sin k\varphi,\ -a\cos k\varphi,\ 0) \quad , \quad r = 0,\ \theta = \frac{\pi}{2} \\ \\ l = \frac{ka^2}{4\pi}\iint \frac{dr d\varphi}{(a^2+r^2)^{\frac{3}{2}}} = \frac{ka}{2|a|}\int\limits_{-\infty}^{\infty} \frac{dr'}{(1+r'^2)^{\frac{3}{2}}} = \frac{ka}{|a|} \end{array} \tag{23.33}$$

It apparently counts how many times the circle of the ring singularity winds around $\overrightarrow{x'}(r)$, the ρ-closed curve of $SU(2)$.

This theoretical reasoning, based on $\pi_2(S^3) = \mathbb{Z}$, needs to be accompanied with the reason why the observed leptonic generations are three and not infinite as it is suggested. The above computations are done in zero gravity limit. Hence the restriction of the number of generations should be caused by gravity through the following reasoning.

The gravitational dressing of the elementary particles satisfies Einstein's equations through the metric $g_{\mu\nu}$, which admits geodetic and shear-free null congruences. These congruences are principal null directions of the Weyl tensor which is formally written

$$\ell^\mu = \overline{\kappa}^{A'}\sigma^b_{A'A}\kappa^A e_b^{\cdot\mu} \quad , \quad \Psi_{ABCD}\kappa^A\kappa^B\kappa^C\kappa^D = 0 \tag{23.34}$$

in the Newman-Penrose formalism. Hence the number of gravitational principal directions cannot exceed four and in the zero gravity approximation we will have

$$\begin{array}{l} e_b^{\cdot\mu} \simeq \delta_b^\mu + O(G) \quad , \quad \kappa^A \simeq \lambda^A(x) + O(G) \\ \Psi_{ABCD} \simeq 0 + G\Psi^0_{ABCD} + O(G^2) \end{array} \tag{23.35}$$

Apparently this general result is going to impose a limitation to the number of the leptonic generations implied in PCFT.

24 COLORED DISTRIBUTIONAL SOLITONS

Concerning the electromagnetic and weak interactions, the hadronic sector of the elementary particles is (about) a copy of the leptonic sector. Quarks simply have the additional strong interaction, which should provide a confining mechanism. The standard model does not explain the general copy-picture, while the artificial add-on of the $SU(3)$ gauge group gives some answers to some phenomena, but it fails to imply (in the continuum) confinement and chirality breaking, which are the characteristic properties of strong interactions.

PCFT is mathematically a principal bundle (gauge field) over a lorentzian CR-manifold. The gluon field is identified with the gauge field of the action and the LCR-structure describes (contains) gravity, electromagnetic and weak interactions as outlined in the previous sections of Part III, where we have assumed that the found distributional solitons (the leptons) have vanishing gluon field configuration (dressing). In this section I will explicitly find stable gluonic configurations for the electron and the neutrino LCR-manifolds, which I will identify with down and up quarks. That is, the origin of the observed general copy-picture between the leptons and quarks is simply their common LCR-structure (which contains gravitational, electromagnetic and weak interactions).

Variation of the actions (5.8) relative to the gauge field implies the field equations

$$\begin{array}{lll}
I_R & \to & \frac{1}{\sqrt{-g}}(D_\mu)_{ij}(\sqrt{-g}(\Gamma^{\mu\nu\rho\sigma}-\overline{\Gamma^{\mu\nu\rho\sigma}})F_{j\rho\sigma})=0 \\
I_I & \to & \frac{1}{\sqrt{-g}}(D_\mu)_{ij}(\sqrt{-g}(\Gamma^{\mu\nu\rho\sigma}+\overline{\Gamma^{\mu\nu\rho\sigma}})F_{j\rho\sigma})=0
\end{array}$$

$$\begin{array}{l}
\Gamma^{\mu\nu\rho\sigma}=\frac{1}{2}[(\ell^\mu m^\nu-\ell^\nu m^\mu)(n^\rho\overline{m}^\sigma-n^\sigma\overline{m}^\rho)+(n^\mu\overline{m}^\nu-n^\nu\overline{m}^\mu)(\ell^\rho m^\sigma-\ell^\sigma m^\rho)] \\
(D_\mu)_{ij}=\delta_{ij}\partial_\mu-\gamma f_{ikj}A_{k\mu}
\end{array}
\tag{24.1}$$

Recall that the derivation of quantum electrodynamics (as an affective field theory) was triggered by the existence of a source in the closed self-dual antisymmetric tensor of the massive static soliton. But, once we assume one of the two actions, the corresponding field equation from the above (24.1) two is exact. We cannot replace (ad hoc) the zero of the second part of the equation with a source, because the symmetries of the action will be destroyed, and subsequently the renormalizability of the action will be destroyed too. The solution to this obstruction comes after a close look at the form of the field equations. Notice that they are the sum or difference of two complex conjugate terms. This does not permit us to apply the complexification (necessary for the application of the Frobenius theorem) and use the convenient form that the LCR-structure tetrad takes in the ambient complex manifold.

Therefore I find convenient to give the PDEs (24.1) the following equivalent forms

$$
\begin{aligned}
I_R \quad &\rightarrow \quad \tfrac{1}{\sqrt{-g}}(D_\mu)_{ij}\{\sqrt{-g}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho \overline{m}^\sigma F_{j\rho\sigma})+ \\
&\qquad +(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\ell^\rho m^\sigma F_{j\rho\sigma})]\} = -k_i^\nu \\
I_I \quad &\rightarrow \quad \tfrac{1}{\sqrt{-g}}(D_\mu)_{ij}\{\sqrt{-g}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho \overline{m}^\sigma F_{j\rho\sigma})+ \\
&\qquad +(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\ell^\rho m^\sigma F_{j\rho\sigma})]\} = -ik_i^\nu
\end{aligned}
\tag{24.2}
$$

where $k_i^\nu(x)$ is a real vector field. That is, taking into account that the sum of the two terms is self-dual

$$
\begin{aligned}
&\Gamma^{\mu\nu\rho\sigma} F_{j\rho\sigma} =: G_j^{\mu\nu} - i * G_j^{\mu\nu} \\
&\Gamma^{\mu\nu\rho\sigma} = \tfrac{1}{2}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho \overline{m}^\sigma - n^\sigma \overline{m}^\rho) + (n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\ell^\rho m^\sigma - \ell^\sigma m^\rho)]
\end{aligned}
\tag{24.3}
$$

the action I_R implies that k_i^ν is a source of the real part $G_j^{\mu\nu}$, while the action I_I implies that k_i^ν is a source of its dual part. The PDEs look like the equations of a gauge field with a color-electric and color-magnetic source respectively. Notice the essential difference of the present equations (24.2) and the conventional gluonic equations. The covariant gauge field derivative $(D_\mu)_{ij}$ applies only on the tetrad-Weyl invariant part of the gauge field and **not** over the entire gauge field. I will solve these partial differential equations in the static (electron) LCR-structure. This is possible, because the LCR-structure defining equations completely decouple from the gauge field equations. The LCR-structure is first fixed (via the Lagrange multipliers) and after we proceed to the solution of the field equations, which involve the gauge field. This property of PCFT is essentially behind the physical observation of the lepton-quark correspondence! That is, a quark has the same LCR-structure with the corresponding lepton. But the quark has in addition a stable non-vanishing distributional gauge field configuration "dressing" (from which it gets its color), while the lepton has vanishing gauge field (gluonic) "dressing".

Recall that a distribution has two parts. The singular part and the regular part. A classical solution of the gauge field with a singular compact source will be interpreted as a colored soliton (the quark) with its gluon potential "dressing" being the regular part of the generalized function outside the singularity region. If we apply again with the gauge covariant derivative $(D_\nu)_{ij}$ and use the commutation relation

$$
[(D_\mu), (D_\nu)]_{ik} = -\gamma f_{ijk} F_{j\mu\nu}
\tag{24.4}
$$

we find that the current must be gauge covariantly "conserved" $(D_\nu)_{ij} k_j^\nu = 0$ for a classical solution to exist (because the group structure constants f_{ijk} are antisymmetric). We will look for fundamental distributional solutions which have compact singular sources, which may be interpreted as localized "particles". I will work out the derivation of a distributional solution for the first PDE (action I_R), where such a solution can exist, and I will simply indicate why the second PDE (action I_I) does not admit a corresponding color-magnetic solution.

In the case of gravity and electromagnetism we found distributional (fundamental) solutions, where the singular part is compact and located at the ring singularity. It is identified with the electron, while its gravitational and electromagnetic fields are the regular parts of the distributions (the gravitational and electromagnetic dressings) located outside the singular support of the source (the electron). Now we will apply the same point of view for the computation of the quark and its gluonic field strength dressing. Outside the compact singular support of color sources, the current $k_j^\nu = 0$ vanishes. In this region we can make the complexification of the real coordinate variable x of the (real) LCR-manifold and after we can make an holomorphic transformation to the LCR-structure coordinates $(z^\alpha(x), z^{\widetilde{\alpha}}(x))$, and use their following powerful properties

$$
\begin{array}{l}
dz^\alpha = f_0^\alpha\ \ell_\mu dx^\mu + f_1^\alpha\ m_\mu dx^\mu \quad , \quad dz^{\widetilde{\alpha}} = f_{\widetilde{0}}^{\widetilde{\alpha}}\ n_\mu dx^\mu + f_{\widetilde{1}}^{\widetilde{\alpha}}\ \widetilde{m}_\mu dx^\mu \\
\\
\ell_\mu dx^\mu = \ell_\alpha dz^\alpha\ ,\ m_\mu dx^\mu = m_\alpha dz^\alpha\ ,\ n_\mu dx^\mu = n_{\widetilde{\alpha}} dz^{\widetilde{\alpha}}\ ,\ \overline{m}_\mu dx^\mu = \widetilde{m}_{\widetilde{\alpha}} dz^{\widetilde{\alpha}} \\
\ell^\mu \partial_\mu = \ell^{\widetilde{\alpha}} \partial_{\widetilde{\alpha}}\ ,\ m^\mu \partial_\mu = m^{\widetilde{\alpha}} \partial_{\widetilde{\alpha}}\ ,\ n^\mu \partial_\mu = n^\alpha \partial_\alpha\ ,\ \overline{m}^\mu \partial_\mu = \widetilde{m}^\alpha \partial_\alpha
\end{array}
\tag{24.5}
$$

In these complex coordinates, the metric takes the off-diagonal form and

$$
\sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = -i\ell \wedge m \wedge n \wedge \widetilde{m} = -i\widehat{g} dz^0 \wedge dz^1 \wedge dz^{\widehat{0}} \wedge dz^{\widehat{1}}
$$

$$
\begin{array}{l}
g_{ab} = \begin{pmatrix} 0 & \widehat{g}_{\alpha\widetilde{\beta}} \\ \widehat{g}_{\beta\widetilde{\alpha}} & 0 \end{pmatrix} \quad , \quad g^{ab} = \begin{pmatrix} 0 & \widehat{g}^{\alpha\widetilde{\beta}} \\ \widehat{g}^{\beta\widetilde{\alpha}} & 0 \end{pmatrix} \\
\widehat{g}_{\alpha\widetilde{\beta}} = \ell_\alpha n_{\widetilde{\beta}} - m_\alpha \overline{m}_{\widetilde{\beta}} \quad , \quad \widehat{g}^{\alpha\widetilde{\beta}} = n^\alpha \ell^{\widetilde{\beta}} - \overline{m}^\alpha m^{\widetilde{\beta}} \quad , \quad \widehat{g} \equiv \det \widehat{g}_{\alpha\widetilde{\beta}} \\
(\ell_0 m_1 - m_0 \ell_1)(n_{\widetilde{0}} \overline{m}_{\widetilde{1}} - \overline{m}_{\widetilde{0}} n_{\widetilde{1}}) = -\widehat{g}\ ,\ (\ell^{\widetilde{0}} m^{\widetilde{1}} - m^{\widetilde{0}} \ell^{\widetilde{1}})(n^0 \overline{m}^1 - \overline{m}^0 n^1) = -\frac{1}{\widehat{g}}
\end{array}
\tag{24.6}
$$

Hence after the complexification we have to replace $\sqrt{-g} \to -i\widehat{g}$. Notice, that now we deal with a complex metric (pseudo-metric), and we must not take complex conjugations before returning back to real x. Then (24.2) takes the form

$$
\begin{array}{lcl}
For\ b = 0 & , & \partial_1 F_{i\widetilde{0}\widetilde{1}} - \gamma f_{ikj} A_{k1} F_{j\widetilde{0}\widetilde{1}} = (D_1)_{ij} F_{j\widetilde{0}\widetilde{1}} = -\widehat{g} k_i^0 \\
For\ b = 1 & , & \partial_0 F_{i\widetilde{0}\widetilde{1}} - \gamma f_{ikj} A_{k0} F_{j\widetilde{0}\widetilde{1}} = (D_0)_{ij} F_{j\widetilde{0}\widetilde{1}} = \widehat{g} k_i^1 \\
For\ b = \widetilde{0} & , & \partial_{\widetilde{1}} F_{i01} - \gamma f_{ikj} A_{k\widetilde{1}} F_{j01} = (D_{\widetilde{1}})_{ij} F_{j01} = -\widehat{g} k_i^{\widetilde{0}} \\
For\ b = \widetilde{1} & , & \partial_{\widetilde{0}} F_{i01} - \gamma f_{ikj} A_{k\widetilde{0}} F_{j01} = (D_{\widetilde{0}})_{ij} F_{j01} = \widehat{g} k_i^{\widetilde{1}}
\end{array}
\tag{24.7}
$$

written separately for every structure coordinate in order to help a non-familiar reader to understand the subsequent mathematical operations. The integrability conditions imply

$$
\begin{array}{l}
[(D_0), (D_1)]_{ik} F_{k\widetilde{0}\widetilde{1}} = -\gamma f_{ijk} F_{j01} F_{k\widetilde{0}\widetilde{1}} = -(D_\alpha)_{ij} (\widehat{g} k_j^\alpha) \\
[(D_{\widetilde{0}}), (D_{\widetilde{1}})]_{ik} F_{k01} = -\gamma f_{ijk} F_{j\widetilde{0}\widetilde{1}} F_{k01} = -(D_{\widetilde{\alpha}})_{ij} (\widehat{g} k_j^{\widetilde{\alpha}})
\end{array}
\tag{24.8}
$$

They vanish outside the compact singular gluonic source.

As expected, the written in LCR-structure coordinates equations do not contain the complexified "metric" $g_{\alpha\widetilde{\beta}}$, and contain only the self-dual left-hand component F_{j01} and right-hand component $F_{j\widetilde{0}\widetilde{1}}$ of the gauge field strength, because the present gauge field action has been constructed to be metric independent.

It is evident that if

$$f_{ijk}F_{j01}F_{k\widetilde{0}\widetilde{1}} = -\frac{1}{g}f_{ijk}(n^{\mu}\widetilde{m}^{\nu}F_{j\mu\nu})(\ell^{\mu}m^{\nu}F_{k\mu\nu}) \neq 0 \tag{24.9}$$

does not vanish outside the sources, the differential equations (24.8) do not admit (fundamental) solutions with compact sources. Hence my conclusion is that, outside the singular compact part (the particle location) of the generalized function, we may have solutions only if an effective abelianization of the partial differential equations (24.8) is achieved. We actually have two types of distributional solutions

$$\begin{array}{ll} Null: & F_{j\widetilde{0}\widetilde{1}} = 0 \quad or \quad F_{j01} = 0 \quad , \quad \forall j \\ \\ Non-null: & (F_{j01} = 0 \quad , \quad F_{k\widetilde{0}\widetilde{1}} = 0) \ , \quad \forall f_{ijk} \neq 0 \\ For\ SU(3): & (F_{j01} = 0 = F_{j\widetilde{0}\widetilde{1}} \ , \ \forall j \neq 3) \ , \ (F_{j01} = 0 = F_{j\widetilde{0}\widetilde{1}} \ , \ \forall j \neq 8) \end{array} \tag{24.10}$$

where the case of the physically interesting group $SU(3)$ has been indicated. These two kinds of solitons will be explicitly computed in the following subsections, while other possibilities of abelianization may also be proposed.

In the above procedure we first found the conditions for the existence of distributional solutions. These conditions impose forms of abelianization while the $SU(3)$ color group is imposed by hand.

24.1 Null colored distributional solitons

In this subsection we will compute the null solutions for the static electron LCR-structure. That is, we will the following two independent solutions

$$\begin{array}{l} A_{\alpha} = \frac{1}{\gamma}(\partial_{\alpha}U)U^{-1} \quad , \quad (\ell^{\mu}m^{\nu}F_{k\mu\nu}) = (\ell^{\widetilde{0}}m^{\widetilde{1}} - \ell^{\widetilde{1}}m^{\widetilde{0}})F_{k\widetilde{0}\widetilde{1}} \neq 0 \\ \\ (n^{\mu}\widetilde{m}^{\nu}F_{k\mu\nu}) = (n^{0}\widetilde{m}^{1} - n^{1}\widetilde{m}^{0})F_{k01} \neq 0 \quad , \quad A_{\widetilde{\alpha}} = \frac{1}{\gamma}(\partial_{\widetilde{\alpha}}U')U'^{-1} \end{array} \tag{24.11}$$

where U and U' are arbitrary elements of the gauge group in a prescribed gauge group representation.

Hence, the two gauge field equations become abelian

$$\begin{array}{l} \partial_{\widetilde{\alpha}}F_{01} - \gamma[A_{\widetilde{\alpha}}, F_{01}] = 0 \quad \Rightarrow \quad \partial_{\widetilde{\alpha}}F'_{01} = 0 \quad , \quad F_{01} = U'F'_{01}U'^{-1} \\ \\ \partial_{\alpha}F_{\widetilde{0}\widetilde{1}} - \gamma[A_{\alpha}, F_{\widetilde{0}\widetilde{1}}] = 0 \quad \Rightarrow \quad \partial_{\alpha}F'_{\widetilde{0}\widetilde{1}} = 0 \quad , \quad F_{\widetilde{0}\widetilde{1}} = UF'_{\widetilde{0}\widetilde{1}}U^{-1} \end{array} \tag{24.12}$$

which apparently coincide with the (abelian) equations

$$\frac{1}{\sqrt{-g}}\partial_\mu\{\sqrt{-g}(\ell^\mu m^\nu-\ell^\nu m^\mu)(n^\rho\overline{m}^\sigma F_{j\rho\sigma})\}=-k_j^\nu \quad , \quad \ell^\mu m^\nu F_{j_{\mu\nu}}=0$$

$$n^\mu\overline{m}^\nu F_{j_{\mu\nu}}=0 \quad , \quad \frac{1}{\sqrt{-g}}\partial_\mu\{\sqrt{-g}(n^\mu\overline{m}^\nu-n^\nu\overline{m}^\mu)(\ell^\rho m^\sigma F_{j\rho\sigma})\}=-k_i^\nu \tag{24.13}$$

Notice that the essential non-vanishing term in both solutions is null, therefore we will look for completely null solutions, i.e. $(\ell^\rho n^\sigma - m^\rho\overline{m}^\sigma)F_{j\rho\sigma}=0$. Hence we will look for null abelian solutions which satisfy the above equations written with differential forms

$$d\{\ell\wedge m(n^\rho\overline{m}^\sigma F_{j\rho\sigma})\}=i*k_j \quad , \quad \ell^\mu m^\nu F_{j_{\mu\nu}}=0 \quad , \quad (\ell^\rho n^\sigma-m^\rho\overline{m}^\sigma)F_{j\rho\sigma}=0$$

$$n^\mu\overline{m}^\nu F_{j_{\mu\nu}}=0 \quad , \quad d\{(n\wedge\overline{m}(\ell^\rho m^\sigma F_{j\rho\sigma})\}=i*k_i^\nu \quad , \quad (\ell^\rho n^\sigma-m^\rho\overline{m}^\sigma)F_{j\rho\sigma}=0 \tag{24.14}$$

The relation between the LCR-tetrad and the LCR-structure coordinates implies that any 2-form

$$F=f(z^\beta)dz^0\wedge dz^1 \quad , \quad \widetilde{F}=\widetilde{f}(z^{\widetilde{\beta}})dz^{\widetilde{0}}\wedge dz^{\widetilde{1}} \tag{24.15}$$

in the ambient complex manifold induces the null self-dual 2-forms

$$F|=A\ell\wedge m \quad , \quad \widetilde{F}|=\widetilde{A}n\wedge\overline{m} \tag{24.16}$$

in the LCR-manifold. But these general self-dual 2-forms do not always have real distributional sources. We will show that the static LCR-structure does have a distributional source, and we will compute it. We will make the calculations with the massive flatprint static LCR-structure using with the asymmetric flatprint LCR-tetrad

$$\begin{array}{l}
L_\mu dx^\mu=[dt-dr-a\sin^2\theta d\varphi] \\
N_\mu dx^\mu=\frac{r^2+a^2}{2(r^2+a^2\cos^2\theta)}[dt+\frac{r^2+2a^2\cos^2\theta-a^2}{r^2+a^2}dr-a\sin^2\theta\ d\varphi] \\
M_\mu dx^\mu=\frac{-1}{\sqrt{2}(r+ia\cos\theta)}[-ia\sin\theta\ (dt-dr)+(r^2+a^2\cos^2\theta)d\theta+ \\
\qquad +i\sin\theta(r^2+a^2)d\varphi]
\end{array} \tag{24.17}$$

and its corresponding structure coordinates

$$\begin{array}{ll}
z^0=t-r+ia\cos\theta \quad , & z^1=e^{i\varphi}\tan\frac{\theta}{2} \\
z^{\widetilde{0}}=t+r-ia\cos\theta \quad , & z^{\widetilde{1}}=\frac{r+ia}{r-ia}e^{-i\varphi}\tan\frac{\theta}{2}
\end{array} \tag{24.18}$$

Then

$$\begin{array}{l}
\frac{1}{\sin\theta(r-ia\cos\theta)}L\wedge M=\frac{-1}{\sqrt{2}z^1}dz^0\wedge dz^1= \\
\quad =\frac{-1}{\sqrt{2}}[\frac{1}{\sin\theta}d(t-r)\wedge d\theta+id(t-r)\wedge d\varphi+a\sin\theta d\theta\wedge d\varphi] \\
\\
\frac{(r+ia\cos\theta)}{(r^2+a^2)\sin\theta}N\wedge\overline{M}=\frac{-1}{\sqrt{2}z^{\widetilde{1}}}dz^{\widetilde{0}}\wedge dz^{\widetilde{1}}= \\
\quad =\frac{-1}{2\sqrt{2}}[\frac{-2ia}{(r^2+a^2)}dt\wedge dr+\frac{1}{\sin\theta}d(t+r)\wedge d\theta-id(t+r)\wedge d\varphi+a\sin\theta d\theta\wedge d\varphi]
\end{array} \tag{24.19}$$

Unlike the electromagnetic field, these self-dual 2-forms are not dimensionless, but they do have distributional sources. Therefore, we expect them to have dimensional charges (interaction constant).

The non-vanishing closed 2-forms (with sources) which satisfy the field equations (24.14) are

$$\begin{aligned}
&d\{\ell\wedge m(n^\rho\overline{m}^\sigma F_{j\rho\sigma})\}=d\{\tfrac{C'_j}{\sin\theta(r-ia\cos\theta)}\ell\wedge m\}=i*k'_j\\
&d\{(n\wedge\overline{m}(\ell^\rho m^\sigma F_{j\rho\sigma})\}=d\{\tfrac{C''_j(r-ia\cos\theta)}{(r^2+a^2)\sin\theta}n\wedge\overline{m})\}=i*k''_j
\end{aligned}\tag{24.20}$$

where C'_j and C''_j are proper dimensional constants, which provide **real** distributional charges, derived using Stokes' theorem. In the asymmetric LCR-tetrad the solutions have the explicit forms

$$\begin{aligned}
&\tfrac{C'_j}{\sin\theta(r-ia\cos\theta)}L\wedge M=\tfrac{-C'_j}{\sqrt{2}}[\tfrac{1}{\sin\theta}d(t-r)\wedge d\theta+id(t-r)\wedge d\varphi+a\sin\theta d\theta\wedge d\varphi]=:\\
&\quad=:F'_j-i*F'_j\\
&\tfrac{C''_j(r+ia\cos\theta)}{(r^2+a^2)\sin\theta}N\wedge\overline{M}=\tfrac{-C''_j}{2\sqrt{2}}[\tfrac{-2ia}{(r^2+a^2)}d(t+r)\wedge dr+\tfrac{1}{\sin\theta}d(t+r)\wedge d\theta-id(t+r)\wedge d\varphi-\\
&\quad-\tfrac{2a^2\sin\theta}{(r^2+a^2)}dr\wedge d\theta+a\sin\theta d\theta\wedge d\varphi]=:F''_j-i*F''_j
\end{aligned}\tag{24.21}$$

Apparently the charges of the sources are generated by the terms $d\theta\wedge d\varphi$ contributing to the imaginary part $i*F_j$ of the self-dual 2-forms. Hence the constants must be imaginary for the sources to be real and the original field equations to be satisfied. Notice that they are proportional to the coefficient a (the radius of disk singularity). We finally find the independent left and right solutions

$$\begin{aligned}
&F'_j=\tfrac{\gamma'_j}{4\pi a}[d(t-r)\wedge d\varphi]=d[\tfrac{\gamma_j}{\pi a\sqrt{2}}(t-r)d\varphi)]\\
&*F'_j=\tfrac{\gamma'_j}{4\pi a}[\tfrac{1}{\sin\theta}d(t-r)\wedge d\theta+a\sin\theta d\theta\wedge d\varphi]\\
&F''_j=\tfrac{\gamma''_j}{4\pi a}[-\tfrac{a}{r^2+a^2}dt\wedge dr+d(t+r)\wedge d\varphi]=\\
&\quad=d[\tfrac{-\gamma''_j\sqrt{2}}{\pi a}(t+r)(\tfrac{a}{r^2+a^2}dr+d\varphi)]\\
&*F''_j=\tfrac{\gamma''_j}{4\pi a}[\tfrac{1}{\sin\theta}d(t+r)\wedge d\theta-\tfrac{2a^2\sin\theta}{(r^2+a^2)}dr\wedge d\theta+a\sin\theta d\theta\wedge d\varphi]
\end{aligned}\tag{24.22}$$

with the corresponding potentials been apparent. The field strengths are dimensionless with a in the denominator. This essential difference of the gluonic solutions with the corresponding electromagnetic one will be studied below.

The second quark (of the massless LCR-structure) can be found using the same procedure. Let us consider the first ($[E=-p^3]$) LCR-structure of (21.8) with the corresponding structure coordinates and tetrad

$$[E=-p^3]: \quad X^3 - aX^1 = 0 \quad , \quad X^0 = 0$$

$$\begin{array}{ll} z^0 = x^0 - ia - x^3 - \frac{(x^1)^2+(x^2)^2}{x^0+x^3-ia} \quad , & z^1 = \frac{x^1+ix^2}{x^0+x^3-ia} \\ z^{\widetilde{0}} = x^0 + x^3 - \frac{(x^1)^2+(x^2)^2}{x^0-x^3} \quad , & z^{\widetilde{1}} = \frac{x^1-ix^2}{x^0-x^3} \end{array} \tag{24.23}$$

The two solutions with sources are expected to have the forms

$$\begin{array}{l} F'_j = f_j(z^0, z^1) dz^0 \wedge dz^1 \quad \to \quad dF'_j = -*k'_j \\ F''_j = f_j(z^{\widetilde{0}}, z^{\widetilde{1}}) dz^{\widetilde{0}} \wedge dz^{\widetilde{1}} \quad \to \quad dF''_j = -*k''_j \end{array} \tag{24.24}$$

But the naive Stokes' theorem does not apply. This problem has to be treated in the "unphysical" grassmannian chart. Apparently, we may bypass this difficulty by assuming a mass term and repeat the preceding calculations, in order to experimentally check PCFT.

The second PDE of (24.2), which is implied by the action I_I, may be written as

$$\begin{array}{ll} I_I \quad \to & \frac{1}{\sqrt{-g}}(D_\mu)_{ij}\{\sqrt{-g}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho \overline{m}^\sigma * F_{j\rho\sigma}) + \\ & +(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\ell^\rho m^\sigma * F_{j\rho\sigma})]\} = -ik_i^\nu \end{array} \tag{24.25}$$

because $\ell^{[\rho}m^{\sigma]}$ and $n^{[\rho}\overline{m}^{\sigma]}$are self-dual. This has exactly the form of the first PDE, with the gauge field tensor replaced by its dual. Hence the solutions of the second PDE will be $-*F'_j$ and $-*F''_j$, which is impossible, because they have sources, i.e. $d*F'_j \neq 0 \neq d*F''_j$.

24.2 Non-null colored distributional solitons

The left F_{i01} and right $F_{j\widetilde{0}\widetilde{1}}$ null solutions may coexist in the same region, if they do not vanish only for i and j in the abelian (Cartan) subalgebra. In the physically interesting case of the $su(3)$ Lie algebra this will happen if i and j take the values 3 and 8. In the physically interesting cases, where the only the potential $A_{3\mu}dx^\mu \neq 0$ does not vanish, we will have the non-null distributional solution with F_3 gluonic dressing, which satisfies the partial differential equations

$$\begin{array}{ll} I_R \quad \to & \frac{1}{\sqrt{-g}}\partial_\mu\{\sqrt{-g}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho \overline{m}^\sigma F_{3\rho\sigma}) + \\ & +(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\ell^\rho m^\sigma F_{3\rho\sigma})]\} = -k_3^\nu \\ I_I \quad \to & \frac{1}{\sqrt{-g}}\partial_\mu\{\sqrt{-g}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho \overline{m}^\sigma F_{3\rho\sigma}) + \\ & +(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\ell^\rho m^\sigma F_{3\rho\sigma})]\} = -ik_3^\nu \end{array} \tag{24.26}$$

An analogous abelian non-null solution with F_8 gluonic dressing exists.

It is more convenient to make calculations using the differential forms. In the case of the first action I_R the PDE takes the form

$$d\{(n^\rho \overline{m}^\sigma F_{j\rho\sigma})\ell \wedge m + (\ell^\rho m^\sigma F_{j\rho\sigma}) n \wedge \overline{m}\} = i * k_j^{(e)} \tag{24.27}$$

Using the preceding procedure we find the non-vanishing closed 2-forms (with real sources) in the case of flatprint massive LCR-tetrad

$$d\{\frac{C'_j}{\sin\theta(r-ia\cos\theta)}L\wedge M+\frac{C''_j(r+ia\cos\theta)}{(r^2+a^2)\sin\theta}N\wedge\overline{M}\}=i*k_j^{(e)} \tag{24.28}$$

where C'_j and C''_j with $j=3,8$ are arbitrary complex constants, which are fixed using Stokes' theorem and the reality conditions for gluonic sources. That is, we have

$$\begin{array}{l}(L^\rho M^\sigma F_{j\rho\sigma})=\frac{C''_j(r+ia\cos\theta)}{(r^2+a^2)\sin\theta}\quad,\quad(N^\rho\overline{M}^\sigma F_{j\rho\sigma})=\frac{C'_j}{\sin\theta(r-ia\cos\theta)}\\ F_{j\rho\sigma}=\partial_\rho A_{j\sigma}-\partial_\sigma A_{j\rho}\quad,\quad j=3,8\end{array} \tag{24.29}$$

We assume that

$$[(L^\mu N^\nu-M^\mu\overline{M}^\nu)F_{j\mu\nu}]=0 \tag{24.30}$$

because it cannot be determined by the sources. That is

$$\begin{array}{l}F_{j\rho\sigma}=-\frac{C'_j}{\sin\theta(r-ia\cos\theta)}(L_\rho M_\sigma-L_\sigma M_\sigma)-\frac{C''_j(r+ia\cos\theta)}{(r^2+a^2)\sin\theta}(N_\rho\overline{M}_\sigma-N_\sigma\overline{M}_\sigma)+c.c.\\ F_{j\rho\sigma}=\partial_\rho A_{j\sigma}-\partial_\sigma A_{j\rho}\quad,\quad j=3,8\end{array} \tag{24.31}$$

For the static flatprint LCR-tetrad the solutions have the explicit forms

$$\begin{array}{l}F_j-i*F_j:=-\frac{2C'_j}{\sin\theta(r-ia\cos\theta)}L\wedge M-\frac{2C''_j(r+ia\cos\theta)}{(r^2+a^2)\sin\theta}N\wedge\overline{M}=\\ \quad=\frac{2C'_j+C''_j}{\sqrt{2}}[\frac{-ia}{r^2+a^2}dt\wedge dr+\frac{1}{\sin\theta}dt\wedge d\theta+\frac{a^2\sin\theta}{r^2+a^2}dr\wedge d\theta\\ \quad-idr\wedge d\varphi+a\sin\theta d\theta\wedge d\varphi]+\\ \quad+\frac{2C'_j-C''_j}{\sqrt{2}}[\frac{ia}{(r^2+a^2)}dt\wedge dr+idt\wedge d\varphi-\frac{r^2+a^2\cos^2\theta}{(r^2+a^2)\sin\theta}dr\wedge d\theta]\end{array} \tag{24.32}$$

After a straightforward calculation I find

$$\int\limits_{t,r=const}[\frac{-2C'_j}{\sin\theta(r-ia\cos\theta)}L\wedge M+\frac{-2C''_j(r+ia\cos\theta)}{(r^2+a^2)\sin\theta}N\wedge\overline{M}]=\frac{(2C'_j+C''_j)4\pi a}{\sqrt{2}}=:-i\gamma_j^{(e)} \tag{24.33}$$

which implies that the (dimensionless) constants γ_3 and γ_8 must be real for the sources to be real and the original field equations to be satisfied.

Assuming $2C'_j-C''_j=0$ (because it is not determined by the gluonic charge), the arbitrary constants are completely fixed and the solutions are

$$\begin{array}{l}F_j=\frac{-\gamma_j}{4\pi a}[\frac{a}{r^2+a^2}dt\wedge dr+dr\wedge d\varphi]=\\ \quad=d[\frac{-\gamma_j}{4\pi a}(\tan^{-1}\frac{r}{a}dt+rd\varphi)]\\ *F_j=\frac{\gamma_j}{4\pi}[\frac{1}{a\sin\theta}dt\wedge d\theta+\frac{a\sin\theta}{r^2+a^2}dr\wedge d\theta+\sin\theta d\theta\wedge d\varphi]\end{array} \tag{24.34}$$

where $j = 3, 8$ for the $SU(3)$ color group and the corresponding potentials been apparent.

The second action I_I does not provide a solution, because it is not compatible with the closed differential form

$$\begin{aligned} &F_{j\rho\sigma} = \partial_\rho A_{j\sigma} - \partial_\sigma A_{j\rho} \quad , \quad j = 3, 8 \\ &dF_j = 0 \end{aligned} \tag{24.35}$$

through which the field enters into the action.

It is interesting to compare the electromagnetic dressing potential

$$A = \tfrac{qr^3}{4\pi(r^4+a^2(x^3)^2)}(dx^0 - \tfrac{rx^1-ax^2}{r^2+a^2}dx^1 - \tfrac{rx^2+ax^1}{r^2+a^2}dx^2 - \tfrac{x^3}{r}dx^3) \tag{24.36}$$

with the gluonic dressing potential of the static LCR-structure in cartesian coordinates

$$\begin{aligned} &x^0 = t \\ &x^1 = (r\cos\varphi + a\sin\varphi)\sin\theta \\ &x^2 = (r\sin\varphi - a\cos\varphi)\sin\theta \\ &x^3 = r\cos\theta \\ \\ &r^4 - [(x^1)^2 + (x^2)^2 + (x^3)^2 - a^2]r^2 - a^2(x^3)^2 = 0 \end{aligned} \tag{24.37}$$

with

$$\begin{aligned} &dt = dx^0 \\ &dr = \tfrac{rx^1-ax^2}{r^2+a^2}dx^1 + \tfrac{ax^1+rx^2}{r^2+a^2}dx^2 + \tfrac{x^3}{r}dx^3 \\ &d\theta = \tfrac{x^3(rx^1-ax^2)}{r^2\sqrt{(r^2+a^2)((x^1)^2+(x^2)^2)}}dx^1 + \tfrac{x^3(ax^1+rx^2)}{r^2\sqrt{(r^2+a^2)((x^1)^2+(x^2)^2)}}dx^2 - \tfrac{\sqrt{(x^1)^2+(x^2)^2}}{r\sqrt{r^2+a^2}}dx^3 \\ &d\varphi = -\tfrac{ax^1+rx^2}{r((x^1)^2+(x^2)^2)}dx^1 + \tfrac{rx^1-ax^2}{r((x^1)^2+(x^2)^2)}dx^2 \end{aligned} \tag{24.38}$$

I find

$$\begin{aligned} A_j^{(g)} &= \tfrac{-\gamma_j}{4\pi a}(\tan^{-1}\tfrac{r}{a}dt + rd\varphi) = \\ &= \tfrac{-\gamma_j}{4\pi a}(\tan^{-1}\tfrac{r}{a}dx^0 - \tfrac{ax^1+rx^2}{(x^1)^2+(x^2)^2}dx^1 + \tfrac{rx^1-ax^2}{(x^1)^2+(x^2)^2}dx^2) \\ \\ F_j^{(g)} &= \tfrac{-\gamma_j}{4\pi a}(\tfrac{a}{r^2+a^2}dt\wedge dr + dr\wedge d\varphi) = \\ &= \tfrac{-\gamma_j}{4\pi(r^2+a^2)}[\tfrac{rx^1-ax^2}{r^2+a^2}dx^0\wedge dx^1 + \tfrac{ax^1+rx^2}{r^2+a^2}dx^0\wedge dx^2 + \tfrac{x^3}{r}dx^0\wedge dx^3]+ \\ &+\tfrac{-\gamma_j}{4\pi ar}[dx^1\wedge dx^2 + \tfrac{x^3(ax^1+rx^2)}{r((x^1)^2+(x^2)^2)}dx^1\wedge dx^3 - \tfrac{x^3(rx^1-ax^2)}{r((x^1)^2+(x^2)^2)}dx^2\wedge dx^3] \end{aligned} \tag{24.39}$$

Notice that unlike the electromagnetic and gravitational potentials, the gluonic potential is not a distribution with the ordinary singular ring singularity. It has a line singularity along the z-axis like the Dirac magnetic monopole, but its asymptotic behavior is different. The "electric" and "magnetic" parts are

$$\begin{aligned} A_{j0}^{(g)} &= \frac{-\gamma_j}{4\pi a}(\tan^{-1}\frac{r}{a}) \\ \overrightarrow{A}_j^{(g)} &= \frac{-\gamma_j}{4\pi}[\frac{r}{a}\frac{-x^2dx^1+x^1dx^2}{(x^1)^2+(x^2)^2} - \frac{1}{2}d\ln((x^1)^2+(x^2)^2)] \end{aligned}$$

$$r = \pm\left\{\frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2} + \sqrt{[\frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2}]^2 + a^2(x^3)^2}\right\}^{\frac{1}{2}} \tag{24.40}$$

where the last term of $\overrightarrow{A}_j^{(g)}$ is a singular gauge. The (19.21) relation

$$r = \pm\left\{\frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2} + \sqrt{[\frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2}]^2 + a^2(x^3)^2}\right\}^{\frac{1}{2}} \tag{24.41}$$

implies

$$\begin{array}{lll} x^3 = 0,\ (x^1)^2+(x^2)^2-a^2<0 & \to & r=0 \\ x^3 = 0,\ (x^1)^2+(x^2)^2-a^2>0 & \to & r=\sqrt{(x^1)^2+(x^2)^2-a^2} \\ (x^3)^2 >> (x^1)^2+(x^2)^2 & \to & r \simeq |x^3| \end{array} \tag{24.42}$$

The "magnetic" part of the potential is linear and the "electric" part is a "well-like" potential.

The other important difference is its singularity relative to the spin parameter a at $a = 0$, which does not permit us to treat gluonic interaction like gravitational and weak interactions as we will discuss in Part IV.

24.3 A chiral $SU(3)$ connection

Recall that Einstein was looking for a geometric structure, which could replace the lorentzian metric, and produce all the interactions. His higher dimensional Kaluza-Klein model could not succeed in describing electomagnetism. The same fate had the metric with torsion suggested by E. Cartan. We already showed that the LCR-structure implies the metric structure and the electroweak connection, which are manifestations of the LCR-tetrad. In the previous subsections, we found distributional solutions of the gauge field related to the static LCR-structure. If we also succeed to derive the $SU(3)$ connection from the LCR-structure, Einstein would be justified. Everything (gravity, electromagnetism, weak and strong interactions) are manifestations of the pure geometric LCR-structure, without any additional gauge field. The suggestion of this subsection is that a $SU(3)$ Cartan connection exists, which could provide the distributional solutions found above.

A realizable LCR-structure is based on hypersurfaces of $CP(3)$, which are covariant relative to $SL(4,\mathbb{C})$ transformation. That is $SL(4,\mathbb{C})$ preserves LCR-structure. Following the Griffiths[17] and Griffiths-Harris[19] works we will look for a possible emergence of the gluonic connection from the LCR-structure itself, without any need to introduce the actions (5.8).

We have already showed how the LCR-manifold is lifted to the grassmannian manifold $G(4,2)$ of the lines of $CP(3)$. Let a frame $\{A_0,\ A_1,\ A_2,\ A_3\}$ of $CP(3)$ determined by the corresponding four vectors of $\mathbb{C}^4$, where A_0 determines the point from where the $CP(3)$ defining lines of $\mathbb{C}^4$ pass through. Assuming it to be a unitary basis (13.31), the Cartan moving frame relations are

$$\begin{array}{l} dA_m = \omega_{mn} A_n \quad , \quad d\omega_{mn} = \omega_{ml} \wedge \omega_{ln} \quad , \quad \omega_{mn} = \overline{\omega_{nm}} \\ l, m, n = 0, 1, 2, 3 \end{array} \tag{24.43}$$

ω_{ml} are $SU(4)$ forms. $CP(3)$ is determined by the annihilation of the 1-forms ω_{0i}, (which is a basis of the cotangent space $T^*(CP(3))$) because of the Frobenius theorem satisfying relation

$$d\omega_{0i} = \omega_{00} \wedge \omega_{0i} + \omega_{0j} \wedge \omega_{ji} \tag{24.44}$$

Hence the projection

$$A_0 :\ U(4) \to CP(3) \tag{24.45}$$

gives a principal $U(1) \times U(3)$ fibration with corresponding vector bundles, the line bundle $L_{A_0} = \overrightarrow{OA_0}$ and the universal quotient bundle $Q_{A_0} = \mathbb{C}^4 / L_{A_0}$. A holomorphic curve $Z(\tau) \subset \mathbb{C}^4$ implies the unitary basis lift

$$\begin{array}{l} Z_0(\tau),\ Z_1(\tau),\ Z_2(\tau),\ Z_3(\tau) \\ dZ_0 = \theta_{00} Z_0 + \theta_{0i} Z_i \end{array} \tag{24.46}$$

where θ_{0i} are of type (1,0) forms. In our case the Kerr function generates a Darboux unitary frame

$$\begin{array}{l} \widehat{Z}_0(\tau,s),\ \widehat{Z}_1(\tau,s),\ \widehat{Z}_2(\tau,s),\ \widehat{Z}_3(\tau,s) \\ d\widehat{Z}_0 = \widehat{\theta}_{00} \widehat{Z}_0 + \widehat{\theta}_{0i} \widehat{Z}_i \\ d\widehat{\theta}_{0i} - iA_{ij} \wedge \widehat{\theta}_{0j} = 0 \quad , \quad \overline{A_{ij}} = A_{ji} \\ A = (A_{ij}) = \sum\limits_{I=1}^{8} A_{I\beta} dz^\beta (t_I)_{ij} \quad , \quad [t_I, t_J] = i f_{IJK} t_K \\ \\ G = \partial A - iA \wedge A \ \longrightarrow DG := \ \partial G + iA \wedge G - iG \wedge A = 0 \end{array} \tag{24.47}$$

where the general antihermitian connection has been replaced with the usual hermitian $su(3)$ gauge field $A = (A_{ij})$, t_J are the generators of $SU(3)$, and $\partial(A_{I\beta} dz^\beta) = \frac{\partial A_{I\beta}}{\partial z^\alpha} dz^\alpha \wedge dz^\alpha$. The explicit form of the curvature is

$$\begin{array}{l} G_{I\alpha\beta} = \partial_\alpha A_{I\beta} - \partial_\beta A_{I\alpha} - f_{IJK} A_{J\alpha} A_{K\beta} \\ \\ G_{I01} = \partial_0 A_{I1} - \partial_1 A_{I0} - f_{IJK} A_{J0} A_{K1} \end{array} \tag{24.48}$$

Notice that the Bianchi identity is identically satisfied, because of the two (complex) dimension of the analytic hypersurface.

The vanishing group manifold curvature $G_{I\alpha\beta} = 0$ should correspond to leptons and $G_{I\alpha\beta} \neq 0$ to quarks (hadrons). In the case of an LCR-structure lifted to $CP(3) \times CP(3)$ we have

$$\rho_{11}(\overline{Z^{m1}}, Z^{n1}) = 0 \quad , \quad \rho_{12}\left(\overline{Z^{m1}}, Z^{n2}\right) = 0 \quad , \quad \rho_{22}(\overline{Z^{m2}}, Z^{n2}) = 0$$

$$K(Z^{m1}) = 0 = K(Z^{m2}) \tag{24.49}$$

This set of solutions is apparently invariant under the $U(4)$ transformation.

Recall that a complex point $(z^{\alpha}, z^{\widetilde{\beta}})$ in the 4-dimensional ambient Kaehler manifold is determined by the two complex points z^{α} and $z^{\widetilde{\beta}}$ of the hypersurface of $CP(3)$. Hence the above holomorphic connection adapted to the analytic surface implies the following section

$$\begin{array}{l} A_J = A_{J\alpha}(z^{\beta})dz^{\alpha} + A_{J\widetilde{\alpha}}(z^{\widetilde{\beta}})dz^{\widetilde{\alpha}} \\ \\ G_{I\alpha\beta}(z^{\beta}) = \partial_{\alpha}A_{I\beta} - \partial_{\beta}A_{I\alpha} - f_{IJK}A_{J\alpha}A_{K\beta} \\ G_{I\widetilde{\alpha}\widetilde{\beta}}(z^{\widetilde{\beta}}) = \partial_{\widetilde{\alpha}}A_{I\widetilde{\beta}} - \partial_{\widetilde{\beta}}A_{I\widetilde{\alpha}} - f_{IJK}A_{J\widetilde{\alpha}}A_{K\widetilde{\beta}} \\ G_{I\alpha\widetilde{\beta}} = -f_{IJK}(A_{J\alpha}A_{K\widetilde{\beta}} - A_{K\alpha}A_{J\widetilde{\beta}}) \end{array} \tag{24.50}$$

which is reduced down to the (real) 4-dimensional LCR-manifold

$$\begin{array}{l} A_J = A_{J\alpha}(z^{\beta}(x))\frac{\partial z^{\alpha}}{\partial x^{\mu}}dx^{\mu} + A_{J\widetilde{\alpha}}(z^{\widetilde{\beta}}(x))\frac{dz^{\widetilde{\alpha}}}{\partial x^{\mu}}dx^{\mu} \\ G = dA + A \wedge A \end{array} \tag{24.51}$$

It is not yet clear to me how to relate such a $SU(3)$ gauge field with the observed gluonic one.

Achieving such a framework could provide Einstein's objective to show that all the interactions (observed in nature) have a geometric origin. Besides, the color group is fixed to the observed in nature $SU(3)$ group.

25 "STRUCTURES" IN BOUNDED REALIZATION

We have already found that the LCR-manifold (the spacetime) may mathematically be viewed as a special totally real submanifold of $\mathbb{C}^4$. Its proper projectivization provides its Cartan lift into a boundary of a domain (10.8) in the grassmannian $G(4,2)$. This boundary may be viewed as a deformation (10.22) of the $SU(2,2)$ classical domain. In its unbounded realization determines the affine Poincaré group, which we used to determine the "electron" and "neutrino" LCR-structures. But it was its bounded realization that permited us to properly study the LCR-ray tracing and reveal why the Kerr-Newman manifold with a naked singularity is well defined in the context of PCFT, while it is rejected in general relativity because of the Hawking-Penrose singularity theorems. But in the bounded realization there are some additional mathematical peculiarities, which have to be understood and their consiquences properly estimated.

In the context of the G. Mack analysis of the "unitary representations of the conformal group $SU(2,2)$ with positive energy"[21], the set of the commuting generators in the bounded and the unbounded realizations is **NOT** the same. In the unbounded realization, the electron soliton is determined by its time-translation P^0 and z-rotation. In the bounded realization, the appropriate automorphisms are $H_0 = \frac{1}{2}(P^0 + K^0)$ (which is essentially a τ translation) and z-rotation $H_1 + H_2$ (16.29), which imply the invariant quadratic Kerr polynomial (16.34). Compare the forms of this polynomial (16.34) with the corresponding polynomial K_e (19.41) of the electron LCR-structure

$$K = A_{03}Y^0Y^3 + A_{12}Y^1Y^2 = \frac{A_{03}+A_{12}}{2}(X^0X^1 - X^2X^3) + \frac{A_{03}-A_{12}}{2}(X^1X^2 - X^0X^3)$$

$$K_e = AY^0Y^1 + (A+2B)Y^0Y^3 + (A-2B)Y^1Y^2) + AY^2Y^3 = B(X^1X^2 - X^0X^3) + AX^0X^1 \tag{25.1}$$

These two polynomials are different, as expected, because only the z-rotation is the same. The other imposed automorphisms are different.

25.1 "U(2) electron" LCR-structure

The Kerr-Newman-Taub-NUT tetrad in the **symmetric tetrad** (t, r, θ, φ) coordinates has the form

$$\begin{array}{l}
\ell_\mu dx^\mu = \frac{1}{\Delta}[\Delta dt - \eta\overline{\eta}dr - \Delta p\ d\varphi] \\
n_\mu dx^\mu = \frac{1}{2\eta\overline{\eta}}[\Delta dt + \eta\overline{\eta}dr - \Delta p d\varphi] \\
m_\mu dx^\mu = \frac{-1}{\sqrt{2}\eta}[-ia\sin\theta dt + \eta\overline{\eta}d\theta + i(r^2+a^2+l^2)\sin\theta d\varphi]
\end{array} \tag{25.2}$$

where

$$\begin{array}{l}
\Delta = r^2 + a^2 - l^2 + h \\
p = a\sin^2\theta - 2l\cos\theta \\
\eta = r + i(l + a\cos\theta) \\
\\
h = -2mr + e^2 \\
\rho^2 = \eta\overline{\eta}
\end{array} \tag{25.3}$$

The above form of the tetrad contains the tetrad-Weyl factors, which give the well known Kerr-Newman-Taub-NUT metric. The regularity of the LCR-structure is found by multiplying the vectors of the tetrad with the necessary factors to make the tetrad regular in $\mathbb{R}^4$ and after checking that it is linearly independent. In the present case we take

$$\begin{array}{l}
\ell' = \Delta\ell_\mu dx^\mu = \Delta dt - \eta\overline{\eta}dr - \Delta p\ d\varphi \\
n' = 2\eta\overline{\eta}n_\mu dx^\mu = \Delta dt + \eta\overline{\eta}dr - \Delta p d\varphi \\
m' = -\sqrt{2}\eta m_\mu dx^\mu = -ia\sin\theta dt + \eta\overline{\eta}d\theta + i(r^2+a^2+l^2)\sin\theta d\varphi
\end{array} \tag{25.4}$$

$$\ell' \wedge m' \wedge n' \wedge \overline{m'} = 4i\sin\theta\Delta[r^2 + (l + a\cos\theta)^2]^3 dt \wedge dr \wedge d\theta \wedge d\varphi$$

It is regular if $\Delta \neq 0$ in $\mathbb{R}^4$. That is $(r-m)^2 + a^2 - l^2 + e^2 - m^2 \neq 0$. Hence only a tetrad with **naked singularity** (in the corresponding riemannian terminology) gives a regular LCR-structure. No other scalar quantities count here. Notice the essential difference with the riemannian geometry. This is the reason that it can exist in the present theory, while it cannot exist in general relativity.

It has the following structure coordinates

$$
\begin{aligned}
& z^0 = t - f_0(r) + ia\cos\theta + 2il\ln(\sin\theta) \quad , \quad z^1 = e^{i\varphi}e^{-iaf_1(r)}\tan\tfrac{\theta}{2} \\
& z^{\widetilde{0}} = t + f_0(r) - ia\cos\theta + 2il\ln(\sin\theta) \quad , \quad z^{\widetilde{1}} = e^{-i\varphi}e^{-iaf_1(r)}\tan\tfrac{\theta}{2} \\
& f_0(r) = \int \tfrac{r^2+a^2+l^2}{\Delta}dr \quad , \quad f_1(r) = \int \tfrac{dr}{\Delta}
\end{aligned}
\tag{25.5}
$$

Its differential forms are

$$
\begin{aligned}
& dl = [\tfrac{\sqrt{2}a(l+a\cos\theta)\sin\theta}{\eta\overline{\eta}^2}m + \tfrac{\sqrt{2}a(l+a\cos\theta)\sin\theta}{\eta^2\overline{\eta}}\overline{m}] \wedge \ell - \tfrac{2i(l+a\cos\theta)}{\eta\overline{\eta}}m\wedge\overline{m} \\
& dn = \tfrac{2r\Delta-\eta\overline{\eta}\Delta'}{2\eta^2\overline{\eta}^2}\ell\wedge n - \tfrac{i\Delta(l+a\cos\theta)}{\eta^2\overline{\eta}^2}m\wedge\overline{m} \\
& dm = [-\tfrac{\Delta}{\sqrt{2}\eta\overline{\eta}^2}\ell + \tfrac{i(l+a\cos\theta)\sin\theta}{\eta\overline{\eta}}n - \tfrac{(r+il)\cos\theta+ia}{\sqrt{2}\eta^2\sin\theta}\overline{m}]\wedge m + \tfrac{\sqrt{2}iar\sin\theta}{\eta^2\overline{\eta}}\ell\wedge n
\end{aligned}
\tag{25.6}
$$

which imply the three relative invariants.

The three self-dual 2-forms

$$
\begin{aligned}
& V_1 := \ell\wedge m \\
& V_2 := n\wedge\overline{m} \\
& V_3 := \ell\wedge n - m\wedge\overline{m}
\end{aligned}
\tag{25.7}
$$

determine the following three closed self-dual 2-forms (up to a compact source). V_3 gives the self-dual electromagnetic field

$$
F_3^+ = \tfrac{C_3}{(r-i(l+a\cos\theta))^2}(\ell\wedge n - m\wedge\overline{m})
\tag{25.8}
$$

V_1 and V_2 give the closed 2-forms

$$
\begin{aligned}
F_1^+ = & \tfrac{C_1}{z^1}dz^0\wedge dz^1 = \tfrac{C_1}{\sin\theta}[-\tfrac{ia\sin\theta}{\Delta}dt\wedge dr + dt\wedge d\theta + i\sin\theta dt\wedge d\varphi - \\
& -\tfrac{r^2+(l+a\cos\theta)^2}{\Delta}dr\wedge d\theta - i\tfrac{r^2+a^2+l^2}{\Delta}\sin\theta dr\wedge d\varphi + (a\sin^2\theta - 2l\cos\theta)d\theta\wedge d\varphi \\
F_2^+ = & \tfrac{C_2}{z^{\widetilde{1}}}dz^{\widetilde{0}}\wedge dz^{\widetilde{1}} = \tfrac{C_2}{\sin\theta}[-\tfrac{ia\sin\theta}{\Delta}dt\wedge dr + dt\wedge d\theta - i\sin\theta dt\wedge d\varphi + \\
& +\tfrac{r^2+(l+a\cos\theta)^2}{\Delta}dr\wedge d\theta - i\tfrac{r^2+a^2+l^2}{\Delta}\sin\theta dr\wedge d\varphi + (a\sin^2\theta + 2l\cos\theta)d\theta\wedge d\varphi
\end{aligned}
\tag{25.9}
$$

The apparent raised question is "what could be the interpretation of this static axisymmetric solution?".

26 UNIVERSE AND ELEMENTARY PARTICLES

In the context of general relativity, where the lorentzian metric is the fundamental quantity with matter viewed as external object, the mathematical problems are plagued by essential singularities defined as non-extendable geodesics. I want to point out that we are still at the classical level. The electron is a (distributional) solitonic LCR-structure with gravitational, electromagnetic and weak dressings. As expected, its LCR-rays are not linear in r, because of its potentials. Even at this classical level it has the electron gyromagnetic ratio $g = 2$. Its source is at a complex point like the (technical) photonic gaussian beams in optics, passing from the one side to the other through an aperture at a plane. The aperture of the electron is the glued essential naked singularity of its Kerr-Newman gravitational dressing. Recall that this aperture could not be studied in the context of Einstein's riemannian geometry (Hawking-Penrose singularity theorems), where such naked essential singularities were rejected as "unphysical", through the "censorship hypothesis" of Penrose. In the present Part III of the Research eBook, we made clear that the LCR-structure is more powerful, and able enough to study such singularities.

It is now the time to consider how the "trees" (elementary particles) are arranged in the "forest" (universe). Apparently the bodies of the universe are aggregations of elementary particles (the singularities of leptons and quarks, and the wavefront singularities of photon and graviton waves). Hence our universe is an LCR-manifold which is mathematically described as a special totally real surface (4.13)

$$\begin{array}{l}
\rho_{11}(\overline{z^{\alpha}}, z^{\alpha}) = 0 \quad , \quad \rho_{12}\left(\overline{z^{\alpha}}, z^{\widetilde{\alpha}}\right) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\alpha}}) = 0 \\
\qquad\qquad\qquad\qquad \Updownarrow \\
\rho_{ij} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \overline{\rho_{12}} & \rho_{22} \end{pmatrix} = 0
\end{array} \tag{26.1}$$

of $\mathbb{C}^4$, which is viewed as a hermitian matrix. In order to study the we have to fix the permitted "motions" of the universe LCR-surface. Recall that in ordinary 3-dimensional euclidian space the general coordinate transformations are restricted the affine transformations. In the present case we start with the general LCR-structure transformations (4.14) to put the neighborhood of a regular point p of the surface into a normalized position, i.e. in the regular coordinates (4.15). These coordinates adapted to p are

$$\begin{array}{l}
z'^{\alpha} = f^{\alpha}(z^{\beta}) \quad , \quad z'^{\widetilde{\alpha}} = f^{\widetilde{\alpha}}(z^{\widetilde{\beta}}) \\
\qquad\qquad\qquad\qquad \Downarrow \\
\rho_{ij} = \frac{1}{2i}\begin{pmatrix} z^{0} - \overline{z^{0}} & z^{\widetilde{1}} - \overline{z^{1}} \\ \overline{z^{\widetilde{1}}} - z^{1} & z^{\widetilde{0}} - \overline{z^{\widetilde{0}}} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \overline{\phi_{12}} & \phi_{22} \end{pmatrix} = 0 \\
\phi_{11}(p) = \phi_{22}(p) = \phi_{12}(p) = 0 \quad , \quad d\phi_{11}(p) = d\phi_{22}(p) = d\phi_{12}(p) = 0
\end{array} \tag{26.2}$$

with ϕ_{ij} satisfying the indicated conditions at p and having special dependence on the complex varables. The above form may also be written as

$$\begin{aligned}
&\begin{pmatrix} z^0 & z^{\widetilde{1}} \\ z^1 & z^{\widetilde{0}} \end{pmatrix} = \begin{pmatrix} u & \overline{\zeta} \\ \zeta & v \end{pmatrix} + i \begin{pmatrix} \phi_{11} & \phi_{12} \\ \overline{\phi_{12}} & \phi_{22} \end{pmatrix} \\
&\phi_{11}(p) = \phi_{22}(p) = \phi_{12}(p) = 0 \quad , \quad d\phi_{11}(p) = d\phi_{22}(p) = d\phi_{12}(p) = 0
\end{aligned} \tag{26.3}$$

where u, v are real and ζ is complex, and they may be assumed as coordinates of the surface. It is convenient to identify $\zeta = z^1$ to consider $\rho_{11} = 0$ the Cauchy null surface with $\ell = \frac{\partial}{\partial v}$. The $\rho_{22} = 0$ is also a Cauchy null surface determined by $n = \frac{\partial}{\partial u}$.

As a generic totally real submanifold of $\mathbb{C}^4$, its real analytic points p have neighborhoods[1] where a general holomorphic transformation annihilates $\phi_{ij} = 0$. These transformations $z^b = f^b(r^c)$ do not preserve the LCR-structure! But in the Kaehler manifold with metric and corresponding symplectic form (15.2)

$$ds^2 = 2\frac{\partial^2 \det(\rho_{ij})}{\partial z^a \partial \overline{z^b}} dz^a d\overline{z^b} \quad , \quad \omega = 2i\frac{\partial^2 \det(\rho_{ij})}{\partial z^a \partial \overline{z^b}} dz^a \wedge d\overline{z^b} \tag{26.4}$$

they are permited. The variables r^b may be chosen such that we get the canonical Darboux symplectic form and the Minkowski induced metric

$$\begin{aligned}
&\omega = 2\eta_{ab} dy^a \wedge dx^b \quad , \quad ds^2|_M = \eta_{ab} dx^a dx^b \\
&r^b =: x^b + iy^b
\end{aligned} \tag{26.5}$$

Apparently this holomorphic transformation cannot be generally extended over the entire $\mathbb{C}^4$, where singularities may appear at complex points which will cause the appearance of non-real analytic points in the universe (the real LCR-submanifold).

At the regular analytic points of the universe, the Minkowski coordinates are well defined through the unbounded realization $SU(2,2)$ symmetric classical domain $X^\dagger E_U X = 0$. But this global spacetime is topologically more complicated then $\mathbb{R}^4$. Its bounded realization reveals that the regular points of the universe may be considered as the $\mathbb{R} \times SU(2)$ universal covering manifold with the singularities of the potentials (dressings) located at compact non real analytic submanifolds of $SU(2)$. The compact parameters w of this spacetime do not appear in (26.3) because we assumed that the coordinates vanish z^a vanish at p. In the next Part IV of this research ebook, the potentials will be treated as operator valued tempered distributions of the rigged Hilbert-Fock space of the Poincaré representations. The essential input will be to introduce such operators for the currents using fermionic representations.

The gravitational, electroweak and gluonic dressings of the fermionic elementary particles are simply the regular parts of the corresponding connections related to the LCR-structures of the elementary particles with the corresponding singular parts being their classical location (trajectory). Precisely, the dressings are the locally integrable functions representatives of the corresponding

Schwartz-distributions, which determine their well defined (as distributions) "ladder" of differentiations. The singularities of the dressings determine the location of the elementary particles (leptons) while the regular parts of the dressings are the "classical" potentials of the elementary particles.

Up to now all the problems of general relativity were affronted with the statement "they will be solved by quantum gravity". PCFT reverses the way of thinking. At the local short scale level, the elementary particles are local communicating holes of the two $\mathbb{R}^4$ patches of the $U(2)$ boundary in the **supposed** embedding of the universe in the grassmannian space $G(4,2)$, apparently if the ambient complex manifold (implied by the linearly independent complex moving frame) can be projectively (algebraically) compactified. Let $\rho_{ij}(\overline{z^b}, z^c) = 0$ be the universal embedding special LCR-conditions. If at a point p of the universe with real coordinates x^μ, these functions are real analytic (without wavefront singularities), nothing occurs at this point. At a neighborhood of these points there is[1] an analytic transformation $z^b(r^a)$ such that the structure conditions take the trivial form $r^a - \overline{r^a} = 0$ in the unbounded coordinate system of the $SU(2,2)$ symmetric classical domain. Such are all the points of the universe without matter and its geometric form is $\mathbb{R} \times S^3$, the universal covering of the boundary of the bounded realization of the classical domain. But the holomorphic extension of $z^b(r^a)$ is not everywhere possible, because it has singularities at the elementary particle loci. There, it is not analytic to both sides of the 4-d universe surface $\rho_{ij}(\overline{z^b}, z^c) = 0$. The one-side analyticity is described using Hormander's tempered generalized functions (or equivalently Sato's hyperfunctions). Recall that the generalized functions are described with ladders of derivatives of locally integrable functions. These locally integrable functions are identified with the potentials and their singular locus with their (material) source. The proper study framework of the generalized functions is the rigged Hilbert-Fock space of tempered distributions. The additional observation is that the found distributional solitons just provide the observed elementary particles of the standard model. No additional elementary particles appear, which could be considered as the hypothetical dark matter weakly interactinf particles. Hence the possibility to understand dark energy and dark matter passes through the formal embedding of the LCR-universe in $\mathbb{C}^4$.

26.1 Dark energy and matter in PCFT

At the large galaxial scale the mathematical problem is based differently. The spacetime is mathematically described as a four dimensional lorentzian manifold embedded into the Kaehler manifold (15.2). This does not mean that the ambient Kaehler manifold is "real". It should be considered as a manifestation of the background LCR-structure.

We have already found that the global universe has to be studied in the bounded realization. Disregarding the singularitie (matter), the universe LCR-manifold is identified with the "natural $U(2)$" LCR-structure. Its Kaehler metric (16.14) is reduced to the de Sitter metric ds_S^2 (18.57), which is also conformally equivalent to the Minkowski metric

$$\begin{array}{l} ds_S^2 = (dt)^2 - T_0^2 \cosh^2 \frac{t}{T_0} [(d\rho)^2 - \sin^2 \rho (d\sigma)^2 - \sin^2 \rho \sin^2 \sigma (d\chi)^2] = \\ \quad = T_0^2 \cosh^2 \frac{t}{T_0} [\widehat{\eta}_{\mu\nu} dx^\mu dx^\mu] \\ \tau = 2 \arctan(e^{\frac{t}{T_0}}) \quad , \quad T_0 := \sqrt{\frac{3}{\Lambda}} \end{array} \tag{26.6}$$

but with $\rho \in [0, 2\pi)$. It covers the entire covering spacetime $R \times SU(2)$. It is essentially imposed by the $U(2)$ topology of the spacetime. The dressings (potentials) of the singularities are local deformations which have to be added. This picture seems to provide the cosmological constant needed to describe dark energy.

The origin of dark matter may be analogous. This may be the effect of the second fundamental form of the embedding of the universe LCR-submanifold in $\mathbb{C}^4$.

In order to visualize the embedding consiquences of the universe as a real submanifold of $\mathbb{C}^4$, it is convinient to work in real coordinates. In the Eisenhart book[9] notation, the Gauss-Codazzi equations have the form

$$\begin{array}{l} R_{ijkl} = \sum\limits_{\sigma=0}^{8} e_\sigma (\Omega_{\sigma|ik} \Omega_{\sigma|jl} - \Omega_{\sigma|il} \Omega_{\sigma|jk}) + \overline{R}_{\alpha\beta\gamma\delta} y_{,i}^\alpha y_{,j}^\beta y_{,k}^\gamma y_{,l}^\delta \\ \\ \Omega_{\sigma|ij,k} - \Omega_{\sigma|ik,j} = \sum\limits_{\tau=0}^{8} e_\tau (\mu_{\tau\sigma|k} \Omega_{\tau|ij} - \mu_{\tau\sigma|j} \Omega_{\tau|ik}) + \overline{R}_{\alpha\beta\gamma\delta} y_{,i}^\alpha y_{,j}^\gamma y_{,k}^\delta \xi_\sigma^\beta \end{array} \tag{26.7}$$

where the universe is V_4 with induced metric g_{ij}, and the enveloping space is V_8 with metric $a_{\alpha\beta}$. The latin indices $i, j, k, ...$ take values up to four and the greek indices $\alpha, \beta, \gamma, \mu, ...$ take values up to eight. We see that the induced curvature R_{ijkl} of the surface depends on the second fundamental form $\Omega_{\sigma|ik}$ and the curvature $\overline{R}_{\alpha\beta\gamma\delta}$ of the ambient Kaehler manifold. These relations of the geometric tensors are essentially implied by their precise dependence on the four real embedding functions $\rho_{ij}(\overline{z^b}, z^c)$, and the implied embedding relations (26.3).

One of the most striking effects attributed to "dark matter" comes from the velocities v^j of the stars at different distances from the galactic center. This mathematical problem is studied in paragraph 48, where the following relation (48.7 p.165) of the curve curvatures: $\frac{1}{\rho_a}$ relative to the ambient metric $a_{\alpha\beta}$, $\frac{1}{\rho_g}$ relative to the spacetime metric g_{ij}, and $\frac{1}{R}$ the normal curvature of spacetime determined by the second fundamental forms

$$\frac{e_a}{\rho_a^2} = \frac{e_g}{\rho_g^2} + \frac{e}{R^2} \tag{26.8}$$

where e_a, e_g and e are plus or minus one. Apparently the deviation of the observed trajectory from the the geodetic one indicates that the observed spacetime is not a totally geodetic submanifold.

In the study of the "accererated" electron we saw (20.13) that the observed trajectory $\xi_R^a(t)$ is the real part of the complex trajectory $\xi^a(t)$. Therefore the computations in the Eisenhart book[9] have to be properly adapted (or interpreted). We have to start from the proper complex trajectory in the grassmannian $G(4,2)$ of a ruled surface.

Part IV

ON THE "ORIGIN" OF QUANTUM THEORY

Synopsis: General relativists have already understood that the existence of inevitable Hawking-Penrose singularities is a drawback of the riemannian geometry. The solution to this problem is to change the fundamental structure. The Penrose censorship hypothesis "naked singularities are not permitted in nature" simply throws in the garbage the clear indications that the Kerr-Newman metric is related with the electron, which Ezra Newman called[27] the magic hidden recesses of general relativity. On the other hand, the expectation that the elementary particles are "quantum objects" and all the open problems will be solved when the quantum theory of everything is found turns out to be wrong too. The intimate relation between the LCR-structure and the Schwartz distributions indicates that quantum theory is a way to cope with this large set of generalized functions. This conjecture is essentially in the background of this Part IV of the present Research eBook and provides a raison d'etre of the success of Bogoliubov perturbative causal approach of quantum field theory.

Pseudo-conformal field theory (PCFT) is based on a well defined renormalizable action (see Part I). Besides the fact that the conditions of the fundamental geometric structure (the lorentzian CR-structure) enter the action through Lagrange multipliers, the path-integral quantization may be viewed as a geometric integration (through cobordism) over the lorentzian CR-manifolds. Recall that this was first observed in 2-dimensional Polyakov action. But quantum electrodynamics and its extension, the standard model of electroweak interactions is based on a correspondence between observed particles and quantum fields. I find that the appropriate formalism to derive this picture is the Bogoliubov perturbative (causal) formulation of quantum field theory and its improvements by Epstein-Glaser as presented by Scharf and collaborators. The essential difference of the present derivation is to consider the Bogoliubov axioms as a consequence of the distributional (generalized functions) character of the solitonic configurations identified with the elementary particles. Recall that the ordinary solitons are smooth functions, but the present solitons are generalized functions, viewed as distributional operators of the well defined rigged Hilbert-Fock space (Gelfand triplet) of the tempered distributions. The Bogoliubov S-matrix and the considered "interacting" fields should be viewed as a harmonic expansion of the distributional (classical) LCR-solitons to the unitary Poincaré representations of the **free** particle fields, the basis of the rigged Hilbert-Fock space of the tempered distributions.

27 TWO PATHWAYS TO "DERIVE" QUANTUM THEORY

The formal consequences of PCFT makes clear that it conforms with the suggestion of E. Cartan to Einstein to look for his unified theory in the general framework of the moving frame $(x^\mu; e_b^\nu \partial_\nu)$ formalism. The simple substitution of the riemannian structure with the lorentzian CR-structure provides enough "formalistic space" to incorporate all currently known particles as stable distributional solitons in the context of a renormalizable LCR-structure compatible action. But PCFT also provides the framework to "derive" quantum field theory itself. The two frameworks are the following:

A. Cobordism approach: We assume that the path-integral of PCFT is the starting point and try to find mathematical methods to compute transition amplitudes over LCR-manifolds with initial and final solitonic configurations. That is, to formulate a 4-dimensional "cobordism approach" based on LCR-manifolds and their precise bundles, analogous to that applied on the Polyakov action, which is a 2-dimensional PCFT. Recall the difficulties to make the corresponding 2-dimensional calculations with the $CR(1)|\_|CR(1)$ cobordism, where the two Cauchy-Riemann boundaries correspond to $\rho_1(\overline{z^0}, z^0) = 0$ and $\rho_2(\overline{z^{\widetilde{0}}}, z^{\widetilde{0}}) = 0$. Only the 2-dimensional cylinder $S^1|\_|S^1$ propagator has been computed so far. Therefore the computations of such an approach seem to be very difficult in the required $CR(3)|\_|CR(3)$ cobordism in 4-dimensional PCFT, where the two Cauchy-Riemann boundaries would be $\rho_{11}(\overline{z^\alpha}, z^\beta) = 0$ and $\rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\beta}}) = 0$.

B. Einstein's approach: It is well known the Einstein-Infeld derivation of the classical equations of gravitational motion of many bodies by simply assuming that they are precise singularities of the gravitational field. In the mathematical terminology, the effort to accommodate the existence of non-trivial deRham cohomologies in a Riemannian manifold imply the equations of gravitational motion of bodies. Apparently everything stops there, because we could not derive the existing elementary particles as stable solitons from the riemannian geometry. But in the previous chapters, we saw that the LCR-structure and the implied compatible 4-dimensional action admit (classical) solitonic solutions, which are Schwartz distributions. Hence the substitution of the Einstein riemannian structure with the LCR-structure opens up the possibility to "derive" quantum theory by simply identifying the quantum Hilbert space with the natural rigged Hilbert-Fock space of tempered distributions.

The passage of classical PCFT to quantized PCFT seems to be analogous to the well known "invention" of the non-rational numbers. Pythagora had based his worldview (ARCHE, theory of everything) to the prime numbers. Near the end of his life he realized that the hypotenuse of a right equilateral triangle with unit sides cannot be written as the ratio of integers, i.e. $\sqrt{2}$ is not a rational number. But the length of hypotenuse is a real (hence measurable) quantity and subsequently $\sqrt{2}$ must be a "physical" number. This discovery destroyed his ARCHE-ToE! HE then disclosed his discovery to the first "circle" of his

”stoa” (masonic lodge) and HE ordered them to kill the member who would disclose it. Hippasus was drown because he did it! Today, we do know that the non-rational numbers are determined by the rational ones $\mathbb{Q}$. $\mathbb{R}$ is a completion of $\mathbb{Q}$. An irrational number is approximately computed, i.e. written as an infinite sum of rational numbers. I mention this story (myth?) in order to stress the analogy between the measurement of the physical quantity ”length of the hypotenuse of a right equilateral triangle”, with the more advanced completion of the Schwartz tempered distributions. That is, the locally integrable singular gravitational, electromagnetic, weak and gluonic dressings belong to an extension of the Fourier transformable functions. Like the computation (using digits with decimals) of a real irrational number needs the infinite decimal representation, so the computation of a real field needs a Cauchy series to be ”computed”, i.e. an expansion into a basis of a Hilbert space. That is, PCFT has the geometric point of view of the dynamics of LCR surfaces of $\mathbb{C}^4$, and quantum field theory describes the dynamics of their singular parts (identified with the elementary particles) as irreducible representations of the Poincaré group in the corresponding rigged Hilbert space.

Following Scharf and collaborators, I will show below that quantum electrodynamics is achieved by the Bogoliubov procedure by simply replacing the classical electromagnetic current with the Dirac field current which satisfies the conservation law. This procedure is extended to the leptonic part of the standard model with the ”spontaneous symmetry breaking” and the subsequent conditions between coupling constants and masses implied. Recall that Epstein-Glaser have already shown that renormalizability of electrodynamics is equivalent to the proper definition of the product of theta (step function) distribution with the Wightman distribution in the framework of Bogoliubov causal quantum field theory. In this formalism, the S-matrix in the complete Hilbert space of the Gelfand triplet (rigged Hilbert space) of the tempered distributions is considered as an expansion to Wick monomials of the Poincaré representations of the free distributional solitons of (classical) PCFT. In the following sections, I will review the Stuckelberg-Bogoliubov causal perturbative field theory adapted to the distributional solitonic spectrum of the classical PCFT, indicating the relevant sections of the books of Bogoliubov and Scharf for details.

28 RIGGED HILBERT SPACE

The bra(c)ket formal notions of Dirac and his delta ”function” constitute the natural formalism of quantum mechanics. Its proper mathematical incorporation to quantum field theory is done through the notion of rigged Hilbert space [[2], Chap. 1]. It is the Gelfand triplet

$$\Omega \subset H \subset \Omega^{\dagger} \tag{28.1}$$

where Ω is a normed vector space ($\equiv$ a vector space with a norm $N(f)$), H is a complex Hilbert space ($\equiv$ a vector space with a complex scalar product (f, g)) and $\Omega^{\dagger}$ is the vector space of linear functionals (dual) of Ω. The normed vector space Ω is not complete ($\equiv$ every Cauchy sequence a_n converges, i.e. if $\lim\limits_{n,m}||a_n - a_m|| = 0$ implies $\lim\limits_{n} a_n = a$ in the same space). But Ω is completed in H. A typical example of such a non-complete vector space is the set of rational numbers, with the set of real numbers its completion. It is well known that the dual of a Hilbert space H is the same vector space, i.e. $H^\dagger = H$. Hence the second part ($H \subset \Omega^\dagger$) is implied from the dualization of the first part ($\Omega \subset H$).

In the present context of rapidly decreasing test functions $\tau(x) \in \mathcal{S}(\mathbb{R}^n)$ and their corresponding tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$, $\Omega = \mathcal{S}(\mathbb{R}^n)$ (also called Schwartz space) and its completion is the Hilbert space of square integrable functions $L^2(\mathbb{R}^n)$. After the passage to the distributions through duality, the interesting properties are transferred to the framework of generalized functions. This passage is essential in the case of unbounded operators A on a dense subspace D_A of a Hilbert space H, which cannot be extended on H.

In the context of Bogoliubov axiomatic formulation of quantum field theory the rigged Hilbert space is

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \tag{28.2}$$

The space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ relative to the supremum definition topology. A generalized eigenvector $f \in \mathcal{S}'(\mathbb{R}^n)$ of an operator A is defined by the relation

$$< f, A\phi >= \lambda < f, \phi >, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n), \quad f \in \mathcal{S}'(\mathbb{R}^n) \tag{28.3}$$

The unitary operator U is defined by the relation

$$(U\phi, U\psi) = (U^{-1}\phi, U^{-1}\psi), \quad \phi, \psi \subset L^2(\mathbb{R}^n) \tag{28.4}$$

The self-adjoint (hermitian) operator A is defined by

$$< Af, \psi >:=< f, A\psi >, \quad \psi \subset \mathcal{S}(\mathbb{R}^n) \quad f \in \mathcal{S}'(\mathbb{R}^n) \tag{28.5}$$

The following completeness theorems of the unitary and hermitian operators are derived: a) A unitary operator in a rigged Hilbert space possesses a complete set of generalized eigenvectors corresponding to eigenvalues λ such that $|\lambda| = 1$. b) A hermitian operator in a rigged Hilbert space possesses a complete set of generalized eigenvectors corresponding to real eigenvalues λ. c) The corresponding spectral theorems and the resolution of the identity are valid.

In conventional quantum field theory, the initial quantum Hilbert space is "rigged" to permit the coexistence of the discrete and continuous spectrum. Here we will first create the rigged Hilbert space H_1 of one free scalar particle and after the rigged Fock space of multiple particles. The point is to work in momentum representation, where the following q^j and p^j are hermitian and they satisfy the quantum commutation relations

$$q^j = \sqrt{m^2+p^2}\, i \tfrac{\partial}{\partial p^j} \tfrac{1}{\sqrt{m^2+p^2}} \quad , \quad p^j \quad \Rightarrow \quad [q^i,\ p^j] = i\delta_{ij}$$

$$(\Phi_1, \Psi_1) = \int\limits_{V^+} \overline{\Phi_1}(p)\Psi_1(p)\tfrac{d^3p}{2p^0}, \quad V^+ \equiv \{p^j\ : p^0 = \sqrt{m^2+p^2} \in \mathbb{R}_+\} \tag{28.6}$$

These permit the definition of the creation and annihilation of the **states** of a free particle

$$
\begin{aligned}
&b_j^\dagger = \tfrac{1}{\sqrt{2}}(p^j + iq^j) \quad , \quad b_j = \tfrac{1}{\sqrt{2}}(p^j - iq^j) \\
&[b_i,\ b_j^\dagger] = \delta_{ij}
\end{aligned}
\tag{28.7}
$$

Do not confuse these creation and annihilation operators in the one-particle Hilbert space with the corresponding operators of the multiparticle states of the Fock space, which we will define below. The "fundamental" state Ψ_0, determined by the condition $b_j\Psi_0 = 0$, may be computed

$$
\Phi_0(p) = \sqrt[4]{\tfrac{m^2+p^2}{\pi^3}}\, e^{-\frac{p^2}{2}}
\tag{28.8}
$$

Using the above one-particle-state creation and annihilation operators, a complete orthonormal basis of **one**-particle-states is found through the relations

$$
\Phi_\nu(p) = \tfrac{(b_1^\dagger)^{\nu_1}(b_2^\dagger)^{\nu_2}(b_3^\dagger)^{\nu_3}}{\sqrt{\nu_1!\nu_2!\nu_3!}}\Phi_0(p)
\tag{28.9}
$$

The spanned space is $\mathcal{S}(\mathbb{R}^3)$. These states admit a Fourier transform, where $e^{ip\cdot x} \in \mathcal{S}'(\mathbb{R}^3)$ are the generalized eigenstates of the continuous spectrum of the momentum operator. That is, the essential property of a rigged Hilbert space is its "capacity" to accommodate a vector basis $\Phi_\nu(p)$ with discrete index ν, and a basis $e^{ip\cdot x}$ with continuous index $\overrightarrow{p}$. The multiparticle second quantized rigged Fock space is constructed by taking the symmetrized or antisymmetrized states as usual [[2], Chap. 4]. The comprehensive and complete description of this procedure in the Bogoliubov and collaborators' books (already classical in quantum field theory), makes a longer review unnecessary.

29 BOGOLIUBOV'S QUANTUM FIELD THEORY

The recursive axiomatic formulation of a quantum field theory has been analyzed in the book of N. N. Bogoliubov, A.A. Logunov and I.T. Todorov[2], and the book of N. N. Bogoliubov and D. V. Shirkov[3]. It approaches the axiomatic formulation of a quantum field theory starting from the S-matrix, the introduction of a "switching on and off" test function $c(x) \in [0,1]$ and assuming the following expansion of the S-matrix

$$
S = 1 + \sum_{n\geq 1} \tfrac{1}{n!} \int S_n(x_1, x_2 ... x_n) c(x_1) c(x_2) ... c(x_n) [dx]
\tag{29.1}
$$

where $S_n(x_1, x_2...x_n)$ are generalized operators, which depend on the **free field** operators (the local Poincaré representations of the free particles). That is, the

S-matrix is an operator valued functional in the Fock space of free relativistic particles, and the above formula is the proper expansion of S. Apparently this perturbative expansion needs the existence of a small coupling constant. The imposed axioms are

$$\begin{array}{l} Poincar\acute{e}\ covariance: \quad U_P S_n(x_1, x_2 ... x_n) U_P^\dagger = S_n(Px_1, Px_2 ... Px_n) \\ Unitarity: \quad SS^\dagger = S^\dagger S = 1 \\ Microcausality: \quad \frac{\delta}{\delta c(x)}[\frac{\delta S(c)}{\delta c(y)} S^\dagger(c)] = 0 \quad for \quad x \precsim y \\ Correspondance\ principle: \quad S_1(x) = iL_{int}[\phi(x)] \end{array} \tag{29.2}$$

where $\phi(x)$ denotes the **free particle** fields (distributional operators in the rigged Fock space) and $x \precsim y$ means $x^0 < y^0$ or $(x-y)^2 < 0$. A general solution of these conditions is

$$\begin{array}{l} S = T[\exp(i\mathbf{L}[\phi(x); c(x))] \\ \\ \mathbf{L}[\phi(x); c(x)] = L_{Int}[\phi(x)]c(x) + \sum\limits_{n\geq 1} \frac{1}{n} \int \Lambda_{n+1}(x, x_1 ... x_n) c(x) c(x_1) ... c(x_n) [dx] \end{array} \tag{29.3}$$

where $\Lambda_{n+1}(x, x_1 ... x_n)$ are quasilocal quantities (arbitrary add-ons of generalized functions), which permit the renormalization process. This order by order construction of a finite S-matrix provides a well established algorithm to distinguish renormalizable from non-renormalizable interaction lagrangians. The initial form of the S matrix contains a non-permitted multiplication of time step functions with other distributions. Epstein-Glaser showed that the recursive procedure does not essentially need these non-defined multiplications. Their procedure is essentially equivalent to the differential renormalization, based on the "scaling degree" of the distribution. The book of Scharf [47] combines the Epstein-Glaser remark with the Bogoliubov procedure providing a mathematically self-consistent description of quantum electrodynamics in the well defined context of Schwartz distributions.

The Bogoliubov formalism is derived from the identification of the "physical Hilbert space" with the complete Hilbert space of the tempered distributions and the well defined rigged Hilbert-Fock space of the free quantum field representations of the Poincaré group. The causality condition should be imposed by the analytic extension of the singular part of the distributions in one from the two sides of $\mathbb{R}^4$ in $\mathbb{C}^4$ in the Siegel realization of the $SU(2,2)$ symmetric classical domain (the edge of the wedge theorem). The advantage of the Bogoliubov procedure is that it may be used in the opposite sense. Knowing the (free) Poincaré representations in the rigged Hilbert-Fock space of tempered distributions, they are identified with "free particles" with precise mass and spin. Then they are described with the corresponding free fields, which are used to write down an interaction lagrangian, suggested by the corresponding dynamics. In the present case, the fundamental dynamics is the PCFT and the particles are the solitonic solutions and their corresponding potentials (dressings) which satisfy the wave equations. The suggested interaction takes the place of the "correspondence principle" in the Bogoliubov procedure. In the present case of effective electrodynamics, the suggested interaction is

$$L_{EM} = e : \overline{\psi}\gamma^\mu\psi A_\mu : \tag{29.4}$$

where ψ is the **free** Dirac field (the operator valued distribution of the $(\frac{1}{2},\frac{1}{2})$ representation) and A_μ is the electromagnetic field (the operator valued distribution of the vector representation). The order by order computation introduces counterterms to the action (with up to first order derivatives). If the number of the forms of the counterterms is finite, the action is renormalizable and the model is considered compatible with quantum mechanics, otherwise the whole construction is rejected as inapplicable. The great value of this constructive procedure will appear in its application for the construction of the action of the standard model.

The idea of Scharf [[48], Chap. 1] and collaborators was for any irreducible representation of the Poincaré group to include all the auxiliary ghost (fermionic) fields required to determine the corresponding physical Hilbert space. The Dirac field does not need any additional field, but the free photon field must satisfy the gauge transformation

$$\begin{gathered} A'_\mu(x) \simeq A_\mu(x) + \lambda\partial_\mu u(x) + O(\lambda^2) \quad , \quad \partial^2 u(x) = 0 \\ A'_\mu(x) = e^{-iQ}A_\mu(x)e^{+iQ} \simeq A_\mu(x) - i\lambda[Q, A_\mu(x)] \end{gathered} \tag{29.5}$$

where Q is the generator of the gauge transformation assumed to be nilpotent. It is found to be

$$\begin{gathered} [Q, A_\mu(x)] = i\partial_\mu u(x) \quad , \quad Q^2 = \tfrac{1}{2}\{Q, Q\} = 0 \\ Q = \int_{x^0} d^3x[\partial_\mu A^\mu \partial_0 u - (\partial_0\partial_\mu A^\mu)u] \\ Q^+ = \int_{x^0} d^3x[\partial_\mu A^\mu \partial_0 \widetilde{u} - (\partial_0\partial_\mu A^\mu)\widetilde{u}] \end{gathered} \tag{29.6}$$

where $u(x)$ and $\widetilde{u}(x)$ is the anticommuting ghost pair which fixes the two physical components from the four Lorentz components of the electromagnetic potential $A_\mu(x)$. So the S matrix is properly expanded into free field operators, because it remains unitary and invariant under the gauge transformation order by order. The precise transformation-derivatives are[48]

$$\begin{gathered} d_Q A_\mu := [Q, A_\mu(x)] = i\partial_\mu u \quad , \quad d_Q u = 0 \\ d_Q \widetilde{u} := \{Q, \widetilde{u}\} = -i\partial_\mu A^\mu \end{gathered} \tag{29.7}$$

The starting lagrangian is gauge invariant, because the free Dirac field satisfies the conserved current relation $\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$. Hence the knowledge of the starting free fields provide the S-matrix order by order. We should not care about singularities and counterterms, which are properly managed by the Epstein-Glaser technique. Hence, PCFT predicts the fermionic massive electron with $g = 2$ and its massless photon interaction through its electromagnetic dressing (which belong to precise Lorentz group representations) and their distributional character imposes the application of the Poincaré compatible rigged Hilbert-Fock space technique. All the rest "quantum field" calculations are just mathematically algorithmic! But the background geometric relations imposes conditions.

29.1 Self-consistency conditions

The perturbative approach permits the definition of general dynamical variables through the generating functional introduced considering the formal existence of a "classical" current $J(x)$ for every field $\phi(x)$ of the action. The generating functional $Z_0(J)$ and the connected generating functional are

$$\begin{aligned} &Z_0(J) =< 0|T[\exp\{i\textstyle\int(L_I(x) + \phi(x)J(x))d^4x\}]|0 > \\ &Z_c(J) = -i\ln[Z_0(J)] \end{aligned} \tag{29.8}$$

The general and the connected Green functions are defined taking (formal) functional derivatives of the generating functionals. Through this formal procedure, the symmetries of the action become Ward identities for the Green functions. The anomalies appear as disagreements between the formal and the exact (quantum) computations.

Any field $\phi(x)$ defines a generating field $\Phi(x; J)$ and the Legendre transformation

$$\begin{aligned} &\Phi(x; J) = \tfrac{\delta Z_c(J)}{\delta J(x)} \\ &Z_c(J) \to W(\Phi) = Z_c(J) - \textstyle\int \Phi(x; J)J(x)d^4x \end{aligned} \tag{29.9}$$

In quantum field theory

$$< 0|\phi(x)|0 >= \phi_C + O(\hbar) \tag{29.10}$$

is the vacuum expectation value of the field $\phi(x)$. We should not confuse it with the "classical" dressing of one elementary particle and the corresponding generating field. In the context of the Bogoliubov-Shirkov notation [[3], Chap. VII]

$$\begin{aligned} &\Phi(x; g) = -\tfrac{\delta H(x;g)}{\delta J(x)} = \tfrac{-i}{g(x)}(\tfrac{\delta S}{\delta J(x)}\overset{*}{S})|_{J=0} \\ &H(x; g) := i(\tfrac{\delta S(g)}{\delta g(x)}\overset{*}{S}(g)) \end{aligned} \tag{29.11}$$

where $H(x; g)$ is the "quantum" hamiltonian of the system. The expected relation of a "dressing" potential of the elementary particles in PCFT and the above formalism is

$$
\begin{aligned}
&A_1(x;1) = -\frac{\delta E(J)}{\delta J(x)}|_{J=0} = \frac{-i}{<S>} \overset{*}{\Phi}_1 (\frac{\delta S}{\delta J(x)} \overset{*}{S}) \Phi_1 |_{J=0} \\
&\Phi_1 = (2\pi)^{\frac{3}{2}} a_\nu^+ (\overrightarrow{k}) \Phi_0
\end{aligned}
\tag{29.12}
$$

where Φ_1 is the one-electron state. Notice that the elementary particle has the same initial and final energies and their creation and annihilation operators are outside the time ordering. The physical intuition is that we use the classical current $J(x)$ as a sensor of the potential generated by a particle.

The relativistic field equations are derived from the Bogoliubov approach and all the experimental results are properly computed [[3], Chap. 7]. Hence (29.12) is going to provide precise self-consistency conditions between PCFT and current quantum field theories, which we will describe below. The causal perturbative approach of quantum field theory has provided the transition amplitudes between the free elementary particles (the stable asymptotic LCR-manifolds), but it is practically impossible to sum up the terms. That is, "quantum" perturbative field theory cannot compute the geometric ring singularity of the elementary LCR-manifolds, which determines the particles themselves and the geometry of the background $\mathbb{R} \times S^3$ universe.

30 "QUANTUM" ELECTRODYNAMICS

Quantum electrodynamics describes extremely well current phenomenology. Up to now it was thought that it also solves the electron gyromagnetic ratio $g = 2$. But this is solved even in the first solitonic approximation in the context of PCFT, where the dressings of the electron naked singularity appear through the Kerr-Newman manifold, viewed as an LCR-manifold. The gyromagnetic ratio of the Kerr-Newman manifold was computed by Carter[5] in the context of general relativity. PCFT simply makes spinorial naked singularities (like electron Kerr-Newman manifold) compatible with the fandemental geometric structure, which is now LCR-structure instead of the metric.

We have already found that the electromagnetic dressing of the (static) electron soliton is

$$
\begin{aligned}
&dF_C = 0 \quad , \quad d * F_C = *j_C \quad , \quad d * j_C = 0 \\
&F^C_{\mu\nu} = \partial_\mu A^C_\nu - \partial_\nu A^C_\mu \quad , \quad \partial_\mu F^{\mu\nu}_C = j^\nu_C \quad , \quad \partial_\nu j^\nu_C = 0
\end{aligned}
\tag{30.1}
$$

where the classical conserved current j^ν_C is written using the delta function indicating the localization of the electron (its singularity ring). This distribution does not contain a wavefront singularity. The existence of this electromagnetic density constitutes the breaking mechanism of the tetrad-Weyl symmetry down to the ordinary Weyl symmetry. As a generalized function, the potential A^C_μ is the locally integrable fundamental distribution, which after its differentiations determines the "ladder" of well defined (non-locally integrable) distributions. But a general solution (as a ruled surface of $CP(3)$ corresponding to a general accelerating trajectory in $G(4,2)$) of the differential equation does contain solutions with the wavefront singularities, which constitute the "quantum" modes of the photon and the electron. The simple formal description of this extension may be achieved through the substitution

$$\begin{array}{l} A_\mu = A_\mu^C + A_\mu^Q \quad , \quad j^\mu = j_C^\mu + e\overline{\psi}\gamma^\mu\psi \\[2ex] \partial_\mu F_C^{\mu\nu} = j_C^\nu \quad , \quad \partial_\mu F_Q^{\mu\nu} = e\overline{\psi}\gamma^\nu\psi \\ [\gamma^\mu(i\partial_\mu - eA_\mu^C(x)) - m]\psi = e\gamma^\mu A_\mu^Q\psi \end{array} \tag{30.2}$$

where the last equation is imposed by the conservation of the current. Recall that the static electron LCR-structure (with the electromagnetic and gravitational dressings) admits the t-translation and z-rotation automorphisms. That is, the solitonic configuration, including its dressings, belong to a representation of the Poincaré group. Hence the distributional character of the electron LCR-structure "forces" us to start with fields A and ψ_e in the rigged Fock space of the tempered distributions and equations

$$\begin{array}{l} \partial_\mu F^{\mu\nu} = e : \overline{\psi_e}\gamma^\nu\psi_e : \\ [\gamma^\mu(i\partial_\mu - eA_\mu(x)) - m]\psi_e = 0 \end{array} \tag{30.3}$$

with $A_\mu^C + A_\mu^Q = A_\mu$ unified in one distributional operator. These equations are also self-consistent with the required charge conservation. The causal approach is implied by the Hilbert space framework. The electromagnetic interaction lagrangian takes the form

$$L_{EM} = e : \overline{\psi_e}\gamma^\mu\psi_e : A_\mu \tag{30.4}$$

as the first term of the Bogoliubov inductive procedure. The implied perturbative S-matrix is compatible with the Q gauge charge algorithm[48]. That is, the Bogoliubov procedure in the rigged Fock space of the "free" photon and electron fields is self-consistent with the above initial term. The Epstein-Glaser remark was that the product of step (Heavyside) function distribution and the free field propagators, which appear in the time-ordering notation, can be properly defined in the context of microlocal analysis as the appropriate multiplication of distributions satisfying the causality condition. In fact Epstein-Glaser remark is equivalent to the conventional differential renormalization based on the scaling transformation [[48], Chap. 2]. Additional "Wightman-Bogoliubov axioms" are not needed any more, because the distributional electron solution is already present in "classical" PCFT. Recall that after the Cauchy completion of the rational numbers $\mathbb{Q}$ to the real numbers $\mathbb{R}$, we do not need any additional physical laws (axioms) to compute the Pythagora hypotenuse. We have just to use the proper mathematics which provide the irrational numbers as infinite sum of rational numbers satisfying the Cauchy criterion. But here we have functions, which must be restricted to the Schwartz space completed into the square integrable functions $L^2(\mathbb{R}^3)$, leading to the well known bound states (Hermite polynomials 28.9).

The first term of the effective electron potential in conventional quantum electrodynamics is

$$
\begin{aligned}
&A_{1_\mu}(x;1) \simeq \tfrac{-i}{2}\overset{*}{\Phi}_1 \tfrac{\delta \widehat{S}_2(J)}{\delta J^\mu(x)}\Phi_1|_{J^\mu=0} \\
&\widehat{S}_2(J) = \int T((L_I(x_1)+A_\nu(x_1)J^\nu(x_1))(L_I(x_2)+A_\nu(x_2)J^\nu(x_2))[dx]
\end{aligned}
\tag{30.5}
$$

which becomes

$$
\begin{aligned}
&A_1^\mu(x) \simeq -e\int D_0^c(x-y)\overset{*}{\Phi}_{1p'} : \overline{\psi_e}(y)\gamma^\mu\psi_e(y) : \Phi_{1p}d^4y \\
&\Phi_{1p} = (2\pi)^{\frac{3}{2}}\overset{*}{a}{}_\nu^{+}(\overrightarrow{p})\Phi_0
\end{aligned}
\tag{30.6}
$$

The electromagnetic dressing (19.80) $A = \frac{qr^3}{4\pi(r^4+a^2(x^3)^2)}\ell_\mu dx^\mu$ in cartesian coordinates

$$
\begin{aligned}
&A = \tfrac{qr^3}{4\pi(r^4+a^2(x^3)^2)}(dx^0 - \tfrac{rx^1-ax^2}{r^2+a^2}dx^1 - \tfrac{rx^2+ax^1}{r^2+a^2}dx^2 - \tfrac{x^3}{r}dx^3) \\
&dF = 0 \quad , \quad d*F = -*j_e
\end{aligned}
\tag{30.7}
$$

of the electron LCR-structure has the proper asymptotic charge e and magnetic moment ea, already computed by Carter without any reference to quantum electrodynamics. Besides, all its components are locally integrable functions determining through derivations the "ladder" of the generalized functions. Hence, it strongly suggests the Einstein approach to the derivation of quantum electrodynamics. But (30.7) is singular at the ring with radius a, while the perturbative terms (30.6) are singular at the point $\overrightarrow{x}=0$, which emerge after an expansion of (30.7) and the definition of r in powers of $a=\frac{\hbar}{2m}$. The emergence of the Plank constant $\hbar$ strongly indicates that (30.7) includes the contributions of loop diagrams. But there is a subtlety. The above electromagnetic field is part of the Kerr-Newman solution, therefore its form could emerge in the context of a causal perturbative approach including electromagnetism and Einstein's gravity.

30.1 "Quantum" electrogravity

The Einstein approach is essentially based on the distributional nature of the present elementary particle-solitons. They are not smooth configurations with finite energy. The energy-momentum conservation of the free particle-soliton (electron) is implied by the existence of a metric $g_{\mu\nu}$ in the tetrad-Weyl class $[g_{\mu\nu}; V_{i\mu\nu},\ i=1,2,3]$, which takes the Kerr-Schild ansatz form and hence the gravitational field

$$
\begin{aligned}
&h_{\mu\nu} = 2f(x)L_\mu L_\nu \\
&g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}
\end{aligned}
\tag{30.8}
$$

satisfies linear field equations with compact singular support sources. The initial interaction lagrangian for the causal perturbative field theory is

$$L_I = \frac{k}{2} h^{\mu\nu} : (\overline{\psi_e}\gamma_\mu \partial_\nu \psi_e - (\partial_\nu \overline{\psi_e})\gamma_\mu \psi_e) : \tag{30.9}$$

with the electron free field. The first third degree gravitational term has been computed [[48], Chap. 5].

The use of the ghost complex of Scharf and collaborators to fix the physical Hilbert-Fock subspace provides the natural framework to study the problem. Their observation that the up to third degree identification with the corresponding terms of the expansion of the Einstein-Hilbert action should be expected, because the other gravitational terms contain higher order derivatives, which generate states with negative norm. Hence, this indicates that the Einstein-Hilbert action with cosmological constant could be properly treated in causal perturbative approach. The nilpotent Q gauge charge method (based on the Krein structure)[48] permits the order by order elimination of the unphysical negative norm "states" and assures the non-emergence of "counterterms" with higher order derivatives.

The above analysis strongly indicates that the perturbative causal approach of electrogravity is well defined providing exact results. Bogoliubov-Shirkov book mentions that in conventional "non-renormalizable" lagrangians the infinite number counterterms should sum up to non-local forms. In the present case of gravity and electromagnetism, this non-locality emerges in the Kerr-Newman metric and electromagnetic potential through the r variable

$$\begin{aligned}
& r^4 - [(x^1)^2 + (x^2)^2 + (x^3)^2 - a^2]r^2 - a^2(x^3)^2 = 0 \\
& \qquad\qquad\qquad\qquad \Downarrow \\
& r = \pm \left\{ \frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2} + \sqrt{[\frac{(x^1)^2+(x^2)^2+(x^3)^2-a^2}{2}]^2 + a^2(x^3)^2} \right\}^{\frac{1}{2}}
\end{aligned} \tag{30.10}$$

which vanishes in the entire disk and not just at a point. Notice that the (naive) expansion in powers of $a = \frac{\hbar}{2m}$

$$r \simeq \pm|\overrightarrow{x}| \mp \frac{(x^1)^2+(x^2)^2}{2|\overrightarrow{x}|^3} a^2 + O(a^4) \tag{30.11}$$

the singularity is restricted at $|\overrightarrow{x}| = 0$.

Perturbative causal approach is an expansion of the S-matrix into the rigged Hilbert-Fock space of the free Poincaré representations, the basis of the free fields tempered distributions. But the relativistic interactions of electron and photons with the atoms have to be treated differently. Under the argument[23],[47] of the classical approximation, the rigged Hilbert space is enriched with the atomic bound states. In the context of quantum electrodynamics, it is formally achieved through the introduction of external sources or equivalently of external electromagnetic potentials[3]. The same approach should be applied in the case of relativistic study of positronium.

Concluding this section, I want to point out the great result of PCFT, that the electromagnetic gauge potential coincides with the ℓ_μ cotangent vector of the LCR-manifold, which is also a null vector of its metric. The implied causal perturbative quantum field theory can provide the first terms in the expansion in a, but the series cannot be summed up. Hence, causal perturbative field theory is excellent to compute the evolution of the point singularities of the distributions, but the ring singularity of the electron cannot be revealed without suming up the series. On the other hand, the identification of the electromagnetic potential with ℓ_μ provides a breaking of the tetrad-Weyl symmetry. In the next section we will show the same "phenomenon" to appear in the weak interactions approach in the context of PCFT.

31 "QUANTUM" WEAK INTERACTIONS

The leptons are found as stable distributional lorentzian Cauchy-Riemann (LCR-) manifolds. The free massive (electron) and its massless partner (neutrino) are explicitly computed as the general and developable (23.3) quadratic ruled surfaces of $CP(3)$. The spin and energy-momentum parameters are determined by their linear (Newman) complex trajectory, i.e.

$$\xi^b(\tau) = v^b\tau + ia^b \tag{31.1}$$

where v^b and a^b are real $SO(1,3)$ contravarient vectors.

The ambiguity in the dummy parameter τ needs a normalization. The following "massive" choice

$$\tfrac{d\xi^b(\tau)}{d\tau}\tfrac{d\xi^c(\tau)}{d\tau}\eta_{bc} = 1 \quad \to \quad v^b v^c \eta_{bc} = 1 \tag{31.2}$$

implies the Einstein energy-momentum relation of special relativity $E^2 - (\overrightarrow{p})^2 = m^2$. The same relation we find if we assume the following "general" choice

$$\begin{array}{c} \xi^b(\tau) = p^b\tau + is^b \quad \to \quad \xi^0(\tau) = E\tau \\ \Downarrow \\ \tfrac{d\xi^b(\tau)}{d\tau}\tfrac{d\xi^c(\tau)}{d\tau}\eta_{bc} = m^2 \quad \to \quad E^2 - (\overrightarrow{p})^2 = m^2 \end{array} \tag{31.3}$$

Notice that the second normalization covers both free electron and its neutrino.

The Kerr-Newman manifold describes the electromagnetic and gravitational dressing of the free electron. I have already computed them using the $SL(2,\mathbb{C})$ bundle over the grassmannian space. The two null vectors ℓ and n are the two gravitational and electromagnetic principal null directions of this type D spacetime.

In the case of the accelerated electron we have, in the general normalization

$$\begin{array}{c} \xi^b(\tau) = \xi^b_R(\tau) + is^b \quad , \quad \xi^0(\tau) = E\tau,\ E \in \mathbb{R} \\ \Downarrow \\ \tfrac{d\xi^b(\tau)}{d\tau}\tfrac{d\xi^c(\tau)}{d\tau}\eta_{bc} =: m^2(\tau) \quad \to \quad E^2 - (\tfrac{d\overrightarrow{\xi}_R(\tau)}{d\tau})^2 = m^2(\tau) \end{array} \tag{31.4}$$

which could be found as the mass definition for massive and massless (developable) ruled surfaces. Hence there is a direct relation between the "massive" ruled surface and its reduced "massless" developable surface. Besides, the electron LCR soliton is determined by a left and right chiral spinors λ^{Ai}, which are different from the "vacuum" light-cone LCR-structure which does not have any local singularity. On the other hand, the flatprint neutrino has only its left part different from the vacuum left part, while its right part coincides with the vacuum right part. And all these chiral "particles" admit the correct vanishing Lie derivatives of the massive and massless Poincaré group, which is a fundamental ingredient of "axiomatic" quantum field theory.

The successful application of the causal perturbative theory to build up quantum electrodynamics and its extraordinary experimental verification, suggest us to extend it including the massless neutrino soliton as a left-hand Dirac field $\frac{1-\gamma_5}{2}\psi_\nu$, and all the permitted charged and neutral currents. No neutrino electromagnetic interaction should be introduced or permit it to appear through the inductive procedure. It has already been shown[48] that assuming the existence of all the standard model particles (for every generation separately) the implied standard model lagrangian is a consequence of the Q gauge charge algorithm. Let us now enumerate the fields and the interactions we will consider in the beginning (correspondence principle) of the Bogoliubov procedure, indicating their existence in the context of PCFT:

1) The massive distributional soliton based on the ruled surface of $CP(3)$ with linear trajectory will be represented with the massive Dirac electron field $\psi_e(x)$, which satisfies the free Dirac equation and hence it implies a free massive Dirac propagator in the time ordering term.

2) The corresponding massless developable ruled surface of $CP(3)$ will be represented with the left-hand part of the massless Dirac neutrino field $\frac{1-\gamma_5}{2}\psi_\nu$, which satisfies the free Dirac equation and hence it implies a free massless Dirac propagator in the time ordering term. All the considered currents will contain only the left-hand part of the neutrino field.

3) The potential of the real part of the closed self-dual 2-form will be represented with the massless electromagnetic field $A_\mu(x)$, which implies the corresponding massless propagator. We have already showed that the LCR-structure implies the existence of a $U(2)$ Cartan connection (22.12), which extends the electromagnetic potential. A scalar field (22.15) also emerges from the relative invariants Φ_i of the LCR-structure. These are identified with the observed electroweak $U(2)$ gauge fields and the corresponding currents are assumed to be defined from the electron and its neutrino.

4) Following the procedure Scharf and collaborators [[48], Chap. 3], we include the necessary ghost pairs for every free field in order to apply the gauge symmetry transformations, which fix the physical Hilbert space. The importance of the order by order fixing of the physical Hilbert space assures that the final counterterms do not include higher order derivatives, a well-known source of negative-norm states. I do not repeat this procedure because it is well described in the second book of Scharf[48].

Including all these assumptions in the initial action through the "correspondence principle", the causal perturbative procedure implies a closed lagrangian form only if the well known relations between the coupling constants and the masses of particles are valid. This means that the "internal symmetry" $U(2)$ breaking mechanism, is a consequence of the initial mass difference between the electron and the neutrino.

Let us now turn to the origin of the three elementary particle generations (23.17, 23.23). We have already computed the Hopf invariants of the left and right chiral parts of the flatprint electron and its neutrino. We found that the left parts of the electron and its neutrino are equal to $+1$, justifying the existence of the charged current in the Hilbert-Fock space of the weak interactions. The right chiral parts of the electron and its neutrino have Hopf invariants -1 and 0 respectively. In the case of the flatprint configurations, there is no limitation on the Hopf invariants, and subsequently on the number of LCR-moving frames. But a self-consistent gravitational dressing restricts the number of geodetic and shear-free null congruences up to four, which is the maximum number of gravitational principal null vectors. Hence I think that the limitation to three particle generations is a consequence of the gravity potentials of these solitons which emerge through the Einstein metric $g_{\mu\nu}$. It is well known that the Einstein metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, where e^a_μ are the four linearly independent vectors of Cartan moving frame. They are defined up to a local $SO(1,3)$ transformation $e'^a_\mu = S^a_b e^b_\mu$ which generates and relates the Cartan connection with the ordinary metric $g_{\mu\nu}$ connection. Newman and Penrose have noticed that assuming a null tetrad, the Cartan formalism acquires very useful properties easily applied to the radiation problems. In the Newman-Penrose formalism the LCR-structure coincides with the existence of two geodetic and shear-free null congruences (with $\kappa = \sigma = \lambda = \nu = 0$). Besides, the use of the spinor dyad $(o^A, \imath^B)$ through the relations

$$\begin{aligned}
&\ell_\mu = \bar{o}^{A'} \sigma^a_{A'B} o^B e_{a\mu} \quad , \quad n_\mu = \bar{\imath}^{A'} \sigma^a_{A'B} \imath^B e_{a\mu} \quad , \quad m_\mu = \bar{\imath}^{A'} \sigma^a_{A'B} o^B e_{a\mu} \\
&o^A \imath^B \epsilon_{AB} = o_B \imath^B = 1
\end{aligned} \tag{31.5}$$

imply the spinorial formulation of general relativity. I have already pointed out that a metric does not always admit two geodetic and shear-free congruences. In this case of metrics, using an arbitrary tetrad, the spinor form of the conformal tensor Ψ_{ABCD} can always be defined, and it admits two spinors $(\lambda^{A1}, \lambda^{B2})$, which satisfy the relation $\lambda^A \lambda^B \lambda^C \lambda^D \Psi_{ABCD} = 0$. In the linearized gravity approximation they become the spinors of the first two rows of the homogeneous coordinates of $G(4,2)$. Hence locally, a non-conformally flat metric compatible with a LCR-structure has at most four geodetic and shear-free null congruences, i.e. at most four branches (sheets). Every two of them determine a LCR-structure. From the Petrov classification, we have the types of spacetime with four (type I), three (type II), two double (type D) and a triple (type III) principal null directions. Apparently the electron and the neutrino solitons correspond to type D spacetimes. I discuss the three leptonic generations in Part III of this Research eBook.

Concluding this section, I think that it would be interesting to perturbatively compute the charged and neutral weak dressings, and the Higgs dressing of the electron generation, using the inverse procedure, from "quantum" to the classical solution.

31.1 Self-consistency conditions

The standard model is a well defined and very successful quantum field theory. In order to invert the point of view, and consider it as consequence of the distributional character of the LCR-structure solitons, we have to fix the $U(2)$ connection, find its curvature, and finally apply the Hodge star exterior derivative to fix the sources. The second application of the Hodge star exterior derivative gives the conservation law for these sources. The starting point is the identification of the (22.9) $U(2)$ connection

$$\begin{aligned}
&B = B_{I\mu}dx^\mu t_I = \begin{pmatrix} \ell' & \overline{m'} \\ m' & n' \end{pmatrix} \quad , \quad [t_I, t_J] = iC_{IJK}t_K \\
&F = dB - iB \wedge B \ \longrightarrow DF := \ dF + iB \wedge F - iF \wedge B = 0
\end{aligned} \tag{31.6}$$

with the weak gauge field.

Electrodynamics suggests the following (hermitian) $U(2)$-connection and curvature for the standard model

$$\begin{aligned}
&B = \begin{pmatrix} A & \overline{W} \\ W & Z \end{pmatrix} = \begin{pmatrix} \ell' & \overline{m'} \\ m' & n' \end{pmatrix} \\
&F = dB - iB \wedge B
\end{aligned} \tag{31.7}$$

where the tetrad is chosen with precise tetrad-Weyl factors. In the case of the electron generation we have

$$\begin{aligned}
&B_{0\mu} + \tfrac{1}{2}B_{3\mu} = \ell'_\mu \quad , \quad B_{0\mu} - \tfrac{1}{2}B_{3\mu} = n'_\mu \quad , \quad \tfrac{1}{2}(B_{1\mu} + iB_{2\mu}) = m'_\mu \\
&F_{0\mu\nu} = \partial_\mu B_{0\nu} - \partial_\nu B_{0\mu} \\
&F_{i\mu\nu} = \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu} - \epsilon_{ijk}B_{j\mu}B_{k\nu}
\end{aligned} \tag{31.8}$$

where $(\ell'_\mu, n'_\mu, m'_\mu)$ is the electron LCR-tetrad with the factors(21.32)

$$\begin{aligned}
&\Lambda = \tfrac{qr}{r^2+a^2\cos^2\theta} \\
&N = -\tfrac{qr}{4\pi(r^2+a^2+h)} \\
&M\overline{M} = -\tfrac{qra\cos\theta}{2\pi(r^2+a^2\cos^2\theta)^2}
\end{aligned} \tag{31.9}$$

In order to help the reader clarify the relations between the tetrad-Weyl transformation and the weak $U(2)$ gauge group, I think I have to make a very brief review. The tetrad-Weyl transformations is the symmetry of the fundamental LCR-structure (which replaces the Einstein metric structure) of PCFT. On the other hand the weak $U(2)$ gauge group is a "Cartan lift" implied by the application of the (holomorphic) Frobenius theorem. Recall that the solution is the real submanifold with the cotangent bundle

$$\begin{gathered}
\rho_{11}(\overline{z^{\alpha}}, z^{\alpha}) = 0 \quad , \quad \rho_{12}\left(\overline{z^{\alpha}}, z^{\widetilde{\alpha}}\right) = 0 \quad , \quad \rho_{22}(\overline{z^{\widetilde{\alpha}}}, z^{\widetilde{\alpha}}) = 0 \\
\Downarrow \\
\begin{pmatrix} \ell & \overline{m} \\ m & n \end{pmatrix} = i(\partial - \overline{\partial}) \begin{pmatrix} \rho_{11} & \rho_{12} \\ \overline{\rho_{12}} & \rho_{22} \end{pmatrix}
\end{gathered} \tag{31.10}$$

and its normal bundle $d\rho_{ij}$. Apparently a weak $U(2)$ gauge transformation breaks the tetrad-Weyl symmetry and connects different LCR-structures. Therefore the sources of the gauge field section must generally be identified with the electron-neutrino weak currents.

The Higgs field is related to the "relative invariant" fields Φ_i. Notice that the weak curvature of the "flat" $U(2)$ LCR-manifold (4.33) vanishes. The great success of the standard model could permit us to invert the "self-consistency conditions". That is, to perturbatively compute effective fields and identify them with the gravitational, electromagnetic, weak and Higgs dressings of the electron-neutrino solitonic system. But it is practically impossible to sum up all the terms and find back the dressings and reveal the lepton and neutrino ring singularity!

Recall that in conventional standard model of the leptonic interactions there is no relation between the gauge fields and Einstein's metric. By simply replacing $g_{\mu\nu}$ with the LCR-structure as fundamental geometric structure (the well-known dream of Einstein) not only quantum Einstein's gravity and standard model are derived, but we precisely find the intimate relation (31.8) between them!

32 "QUANTUM" CHROMODYNAMICS

In the context of the fundamental action of PCFT, quantum chromodynamics could be defined by the tetrad-Weyl covariant gauge field action. I computed the gluonic dressing (24.39) of the static axisymmetric LCR-manifold. The leptonic solitons have vanishing gluonic dressing and the corresponding quark have non-vanishing gluonic dressing. This general procedure explains the particle correspondence between leptons and hadrons. Besides the $SU(3)$ group should be determined by an anomaly cancellation, in complete analogy to the computation of the dimension 26 of the ambient spacetime in the Polyakov action (the 2-dimensional PCFT). I have also considered the possibility that the $SU(3)$ gauge group could emerge from a Cartan lift of the LCR-structure. That is, that the gluonic field has a geometric origin like gravity and electroweak gauge fields. The implied gluonic connection (24.50) essentially coincides with the previous one, while the $SU(3)$ gauge group is also fixed.

But the form of the electron electromagnetic dressing and its potential

$$\begin{aligned}
F = & \tfrac{q}{4\pi(r^2+a^2\cos^2\theta)^2}[(r^2-a^2\cos^2\theta)dt\wedge dr - 2a^2 r\cos\theta\sin\theta dt\wedge d\theta+ \\
& +2a^2 r\cos\theta\sin\theta dr\wedge d\theta + a(r^2-a^2\cos^2\theta)\sin^2\theta dr\wedge d\varphi- \\
& -2ar(r^2+a^2)\cos\theta\sin\theta d\theta\wedge d\varphi = \\
& = d[\tfrac{qr}{4\pi(r^2+a^2\cos^2\theta)}(dt-dr-a\sin^2\theta d\varphi)] \\
& \qquad\qquad\qquad\qquad \Downarrow \\
A = & \tfrac{qr^3}{4\pi(r^4+a^2(x^3)^2)}(dx^0 - \tfrac{rx^1-ax^2}{r^2+a^2}dx^1 - \tfrac{rx^2+ax^1}{r^2+a^2}dx^2 - \tfrac{x^3}{r}dx^3)
\end{aligned} \tag{32.1}$$

is completely different from the gluonic dressing of the corresponding quark

$$\begin{aligned}
F_j^{(g)} &= -\tfrac{\gamma_j}{4\pi a}[\tfrac{a}{(r^2+a^2)}dt\wedge dr + dr\wedge d\varphi] = \\
&= d[\tfrac{\gamma_j}{4\pi a}(\tan^{-1}\tfrac{r}{a}dt - rd\varphi)] \\
*F_j^{(g)} &= -\tfrac{\gamma_j}{4\pi}[\tfrac{1}{a\sin\theta}dt\wedge d\theta + \sin\theta d\theta\wedge d\varphi]
\end{aligned} \tag{32.2}$$

Notice that the spin parameter a appears in the denominator which permits the quark gluonic potential to have a magnetic linear component that could provide a kind of confinement. On the other hand the electromagnetic dressing can be expanded in powers of a permitting the appearance of a linearized interaction, while the gluonic dressing is singular at $a = 0$. No self-consistency conditions seem to apply. The form of the "quantum" gluonic interaction is not evident (to me).

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